On initial boundary value problems for systems of parabolic equations

A.Sh. Lyubanova
Siberian Federal University
e-mail: lubanova@mail.ru

The initial boundary value problems for systems of two parabolic equations are studied when the conditions with respect to the time variable are given only for one of the unknown functions. The problems are considered in the case where along with the initial data for one of the functions either the value of the same function is given at the final moment of time or the integral of this function with respect to time is known. The existence and uniqueness of the solution to these problems are established.

1. The preliminaries

Let Ω be a domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$, $\overline{\Omega}$ is the closure of Ω, T is an arbitrary positive real number, $Q_T = (0,T)\times\Omega$ and $S_T = (0,T)\times\partial\Omega$. Throughout this paper we use the notation: $\|\cdot\|_R$ and $(\cdot,\cdot)_R$ are the norm and the inner product of $\mathbf{R}^n$, respectively; $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively; $\|\cdot\|_j$ and $\langle\cdot,\cdot\rangle_j$ are the norm of $W_2^j(\Omega)$ and the duality relation between $\overset{\circ}{W}{}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$, respectively; as usual $W_2^0(\Omega) = L^2(\Omega)$. We also denote by $H^q(\overline{\Omega})$ the Hölder space of functions dependent on the variables $x \in \overline{\Omega}$, $q > 0$.

Let $M, L, B : W_2^1(\Omega) \to (W_2^1(\Omega))^*$ are linear differential operators of the form

$$M = -\mathrm{div}(\mathcal{M}(x)\nabla) + (\mathbf{m}, \nabla)_R + m(x)I, \tag{1.1}$$

$$L = -\mathrm{div}(\mathcal{L}(x)\nabla) + (\mathbf{l}, \nabla)_R + l(x)I, \tag{1.2}$$

$$B = -\mathrm{div}(\mathcal{B}(x)\nabla) + (\mathbf{b}, \nabla)_R + b(x)I, \tag{1.3}$$

respectively, where $\mathcal{M}(x) \equiv (m_{ij}(x))$, $\mathcal{L}(x) \equiv (l_{ij}(x))$ and $\mathcal{B}(x) \equiv (b_{ij}(x))$ are matrices of functions, $i,j = 1,2,\ldots,n$; $\mathbf{m}$, $\mathbf{l}$ and $\mathbf{b}$ are vector functions; m, l, b are scalar functions; I is the identical operator.

We assume that the following conditions are fulfilled.

I. $m_{ij}(x) \in H^{r+1}(\overline{\Omega}) \cap W_\infty^3(\Omega)$, $l_{ij}(x), b_{ij}(x) \in W_\infty^1(\Omega)$, $i,j = 1,2,\cdots,n$, $0 < r < 1$; $\mathbf{m} \in (H^r(\overline{\Omega}) \cap W_\infty^2(\Omega)))^n$, $m(x) \in H^r(\overline{\Omega}) \cap W_\infty^2(\Omega)$; $\mathbf{l}, \mathbf{b} \in (L^\infty(\Omega))^n$, $l(x), b(x) \in L^\infty(\Omega)$.

II. M and L are operators of elliptic type, that is, there exist positive constants m_k and l_k, $k = 1,2$, such that for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$m_1\|v\|_1^2 \le \langle Mv, v\rangle_1 \le m_2\|v\|_1^2, \tag{1.4}$$

$$l_1\|v\|_1^2 \le \langle Lv, v\rangle_1 \le l_2\|v\|_1^2. \tag{1.5}$$

In this paper we are studying the following problems.

Problem 1. For given functions $g_k(x)$, $f_k(t,x)$, $\beta_k(t,x)$, $k=1,2$, $\sigma(x,p)$, $u_0(x)$ and $u_T(x)$ find the pair of functions $\{u_1(t,x), u_2(t,x)\}$ satisfying the system of equations

$$u_{1t} + Mu_1 = g_1(x)U_1 + g_2(x)U_2 + f_1(t,x), \tag{1.6}$$

$$u_{2t} + Lu_2 = Bu_1 + \sigma(x,u_1) + f_2(t,x), \qquad (t,x) \in Q_T, \tag{1.7}$$

and the conditions

$$u_1|_{t=0} = u_0(x), \quad u_1|_{t=T} = u_T(x), \qquad x \in \overline{\Omega}, \tag{1.8}$$

$$u_i|_{S_T} = \beta_i(t,x). \tag{1.9}$$

Here $U_i(x) \equiv \int_0^T u_i dt$, $i=1,2$.

Problem 2. For given functions $f_i(t,x)$, $\beta_i(t,x)$, $i=1,2$, $\mu(x)$ and $\varphi(x)$ find the pair of functions $\{u_1(t,x), u_2(t,x)\}$ satisfying the system of equations

$$u_{1t} + Mu_1 = Bu_2 + f_1(t,x), \tag{1.10}$$

$$u_{2t} + Lu_2 = f_2(t,x), \qquad (t,x) \in Q_T, \tag{1.11}$$

the conditions

$$u_1|_{t=T} - u_1|_{t=0} = \mu(x), \quad U_1(x) = \varphi(x), \qquad x \in \overline{\Omega}. \tag{1.12}$$

and the boundary data (1.9).

In addition to Problems 1 and 2 we consider two auxiliary problems for the linear parabolic equation

$$u_t + Lu = F(t,x) \tag{1.13}$$

with the operator L of the form (1.2), the boundary condition

$$u|_{S_T} = \beta(t,x). \tag{1.14}$$

and initial data

$$u|_{t=T} - u|_{t=0} = \mu(x), \qquad x \in \overline{\Omega}, \tag{1.15}$$

or

$$\int_0^T u(t)dt = \varphi(x), \qquad x \in \overline{\Omega}. \tag{1.16}$$

The existence and uniqueness of the solution to the problems (1.13)–(1.15) and (1.13), (1.14), (1.16) is guaranteed by the following theorems.

Theorem 1.1. *Let the assumptions I, II of the operator L be fulfilled. Let $\mu \in L_2(\Omega)$, $F \in L^2(0,T;W_2^{-1}(\Omega))$, $\beta \in L^\infty(0,T;W_2^{1/2}(\partial\Omega))$, $\beta_t \in L^2(S_T)$. Then the problem (1.13) –(1.15) has a unique weak solution u in the class $Y \equiv L^2(0,T;W_2^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and the solution depends continuously on F, μ and β, i.e.,*

$$\|u\|_Y \le C\Big\{\|\mu\| + \|\beta\|_{L^\infty(0,T;W_2^{1/2}(\partial\Omega))} + \|\beta_t\|_{L^2(S_T)} + \|F\|_{L^2(0,T;W_2^{-1}(\Omega))}\Big\}$$

where the positive constant C depends on n, T, l_1, l_2 and mesΩ.

Theorem 1.2. *Let the assumptions I, II of the operator L be fulfilled and $\partial\Omega \in C^2$. Let $F \in L^2(Q_T)$, $\varphi \in W_2^2(\Omega)$, $\beta \in L^\infty(0,T;W_2^{3/2}(\partial\Omega))$, $\beta_t \in L^2(S_T)$. Then the problem (1.13)–(1.14),(1.16) has a unique weak solution u in the class Y and the solution depends continuously on F, μ and β, i.e.*

$$\|u\|_Y \le \tilde{C}\Big\{\|L\varphi\| + \|\beta\|_{L^\infty(0,T;W_2^{1/2}(\partial\Omega))} + \|\beta_t\|_{L^2(S_T)} + \|F\|_{L^2(Q_T)}\Big\}$$

where the positive constant $\tilde{C}$ depends on n, T, l_1, l_2 and mesΩ.

Theorems 1.1 and Theorem 1.2 are reduced to the special cases of Theorems 3 and 7 of [2], respectively, by substitution of the function u with a function $w + \rho$ in (1.13),(1.14),(1.16) where w is a new unknown function, $M\rho = 0$ and $\rho|_{S_T} = \beta$.

2. The problem with the initial and final conditions for u_1

In this section we are interested in finding the sufficient conditions for the existence and uniqueness of a solution to Problem 1. By a solution of Problem 1 we mean the pair $\{u_1, u_2\}$ of the class $V = \{\{u_1, u_2\}|u_1 \in C([0,T]; W_2^4(\Omega)),\ u_{1t} \in L^2(0,T; W_2^4(\Omega)),$ $u_2 \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_2^1(\Omega)),\ u_{2t} \in L^2(0,T; W_2^{-1}(\Omega))\}$ which satisfies (1.6)–(1.9).

Theorem 2.1. *Let the assumptions I, II be fulfilled and $\partial\Omega \in C^4$. Let also*

i) $f_1 \in C^r([0,T]; W_2^2(\Omega) \cap H^r(\overline{\Omega}))$, $f_{1t} \in C([0,T]; W_2^2(\Omega))$, $f_{1tt} \in L^2(Q_T)$, $\beta_1 \in C(0,T; H^{r+2}(\partial\Omega) \cap W_2^{7/2}(\partial\Omega))$, $\beta_{1t} \in C^r([0,T]; H^r(\partial\Omega)) \cap L^2(0,T; W_2^{7/2}(\partial\Omega))$, $\beta_{1tt} \in L^2(0,T; W_2^{3/2}(\partial\Omega))$, $u_0, u_T \in H^{r+2}(\overline{\Omega}) \cap W_2^5(\Omega)$, $0 < r < 1$, $f_2 \in L^2(Q_T)$, $\beta_2 \in L^2(0,T; W_2^{3/2}(\partial\Omega))$;

ii) $g_k(x)$, $k = 1, 2$, *are twice continuously differentiable in* $\overline{\Omega}$, *there exists a constant* $\nu > 0$ *such that* $|g_2(x)| \geq \nu$ *for all* $(x,p) \in \overline{\Omega} \times (-\infty, +\infty)$; $a(x,p)$ *is continuous in* $\overline{\Omega} \times (-\infty, +\infty)$ *and* $\|a(x,v)\| < +\infty$ *for all* $v \in W_2^4(\Omega)$.

Then Problem 1 has a unique solution $\{u_1, u_2\} \in V$ *and* $u_{1tt} \in L^2(0,T; W_2^2(\Omega))$.

Proof. We prove the theorem in two steps. In the first step we establish the existence and uniqueness of the solution to the problem (1.6),(1.8),(1.9) as an inverse problem of recovering an unknown source function U_2. The second step consists of finding u_2 as the solution of the problem for (1.7) with the boundary data (1.9) and proving the uniqueness of u_2 provided that u_1 and U_2 are known.

Step 1. Let us consider the problem (1.6),(1.8),(1.9). We integrate (1.6) with respect to t on $[0,T]$ and divide the result by T. In view of (1.8) this yields

$$\delta u_1 + M\bar{u}_1 = g_1(x)U_1 + g_2(x)U_2 + \bar{f}_1 \tag{2.1}$$

where $\delta u_1 = T^{-1}(u_T(x) - u_0(x))$ and $\bar{v} = T^{-1}\int_0^T v\,dt$ for every $v \in L^1(0,T)$. Subtracting (2.1) from (1.6) we obtain

$$\bar{u}_{1t} + M\bar{u}_1 = f_1 - \bar{f}_1 + \delta u_1.$$

Introducing the function h as a solution of the problem $Mh = 0$, $h|_{\partial\Omega} = \beta_1$ and rewriting the last equality in terms of a new function $w = u_1 - \bar{u}_1 - h + \bar{h}$ we are led to the equation

$$w_t + Mw = f_1 - \bar{f}_1 + \delta u_1 - h_t \equiv F_1(t,x). \tag{2.2}$$

From (1.8) and (1.9)

$$w|_{t=T} - w|_{t=0} = T\delta u_1 - h(T,x) + h(0,x) \equiv w_T(x). \tag{2.3}$$

$$w|_{\partial\Omega} = 0. \tag{2.4}$$

By Theorem 1.1, the problem (2.2)–(2.4) has a unique generalized solution $w \in Y$ and

$$\|w\|_Y \leq C\Big\{\|\delta u_1\| + \|F_1\|_{L^2(0,T;W_2^{-1}(\Omega))}\Big\}.$$

Let us prove that in the hypotheses of the theorem $w \in C(0,T;W_2^4(\Omega))$. To do this we consider the iterative scheme

$$w_t^s + Mw^s = F_1(t,x) \tag{2.5}$$

$$w^s|_{t=0} = w^{s-1}|_{t=T} - \delta u_1. \tag{2.6}$$

$$w^s|_{\partial\Omega} = 0, \quad s = 1, 2, ...; \qquad w^0 = 0. \tag{2.7}$$

By [1, p. 364] the solution $w^s \in C^r([0,T];H^{r+2}(\overline{\Omega}))$ and $w_t^s \in C^r([0,T];H^r(\overline{\Omega}))$. Subtracting (2.5) for w^s from the same equation for w^{s+1} gives

$$w_t^{s+1} - w_t^s + Mw^{s+1} - Mw^s = 0. \tag{2.8}$$

We multiply (2.8) by $M(w^{s+1}-w^s)$ in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the first term of the resulting equation. This yields

$$\frac{1}{2}\frac{d}{dt}\big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1 + \|M(w^{s+1} - w^s)\|^2 = 0.$$

Multiplying this equation by $e^{\alpha t}$ where $\alpha > 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\Big[e^{\alpha t}\big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1\Big] e^{\alpha t}\|M(w^{s+1} - w^s)\|^2 -$$

$$+\alpha e^{\alpha t}\big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1 = 0. \tag{2.9}$$

In view of the ellipticity of the operator M there exists a constant $m_3 > 0$ such that for any $v \in W_2^2(\Omega)$, $v|_{\partial\Omega} = 0$

$$\|Mv\| \geq m_3\|v\|_2. \tag{2.10}$$

Hence

$$\|M(w^{s+1} - w^s)\|^2 - \big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1 \geq (m_3^2 - \alpha m_2)\|w^{s+1} - w^s\|_2^2. \tag{2.11}$$

Choosing $\alpha < m_2^2/m_1$ and integrating (2.9) with respect to t from 0 to τ, $0 < \tau \leq T$, we get

$$\frac{1}{2}\big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1 + (m_3^2 - \alpha m_2)\int_0^\tau \|w^{s+1} - w^s\|_2^2\, e^{-\alpha(\tau - t)}\, dt$$

$$\leq \frac{1}{2}e^{-\alpha\tau}\big\langle w^s - w^{s-1}, M(w^s - w^{s-1})\big\rangle_1\big|_{t=T} \tag{2.12}$$

by (2.10). From (2.11) and (2.12) it follows that

$$\big\langle w^{s+1} - w^s, M(w^{s+1} - w^s)\big\rangle_1\big|_{\tau=T} \leq e^{-\alpha T}\big\langle w^s - w^{s-1}, M(w^s - w^{s-1})\big\rangle_1\big|_{t=T}$$

$$\leq e^{-\alpha T(s-1)}\big\langle w^1, Mw^1\big\rangle_1\big|_{t=T} \leq e^{-\alpha Ts}\big\langle w_T, Mw_T\big\rangle_1 \tag{2.13}$$

and

$$\int_0^T \|w^{s+1} - w^s\|_2^2\, dt \leq (m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)}\big\langle w_T, Mw_T\big\rangle_1. \tag{2.14}$$

for every $s = 1, 2, ...$ Equation (2.8) and the last inequality enable to obtain the estimate for $w_t^{s+1} - w_t^s$. Namely,

$$\int_0^T \|w_t^{s+1} - w_t^s\|^2\, dt \leq m_4(m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)}\big\langle w_T, Mw_T\big\rangle_1. \tag{2.15}$$

Here the positive constant m_4 depends on $\max_{x\in\Omega}\{|m_{ij}(x)|, |\mathbf{m}(x)|, |m(x)|\}$.

Under the hypotheses of the theorem $w_t^s + Mw^s \in W_2^2(\Omega)\cap H^r(\overline{\Omega})$ for every $t \in [0,T]$, $(w_t^s + Mw^s)_t \in L^2(0,T;W_2^2(\Omega))$ and $w^s(0,x), w^s(T,x) \in W_2^2(\Omega)$. Moreover, the equation (2.5) is valid for all $(t,x)\in \overline{Q}_T$. Acting on (2.5) and (2.6) with the operator M we obtain the following equalities for function $W^s = Mw^s - F_1(t,x)$:

$$W_t^s + MW^s = F_{1t}, \tag{2.16}$$

$$W^s|_{t=0} = W^{s-1}|_{t=T} - Mw_T + T\delta F_1 \tag{2.17}$$

where $\delta F_1 = T^{-1}(F_1(T,x) - F_1(0,x))$. From (2.5), (2.7) we have

$$W^s|_{\partial\Omega} = 0. \tag{2.18}$$

The problem (2.16)–(2.18) has a unique solution $W^s \in L^2(0,T;W_2^2(\Omega))$. Repeating the arguments led to (2.14) and (2.15) one can obtain estimates for W^s. Namely,

$$\int_0^T \|W^{s+1} - W^s\|_2^2\,dt \le (m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)} \big\langle Mw_T - T\delta F_1, M(Mw_T - T\delta F_1)\big\rangle_1, \tag{2.19}$$

$$\int_0^T \|W_t^{s+1} - W_t^s\|^2\,dt \le m_4(m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)} \big\langle Mw_T - T\delta F_1, M(Mw_T - T\delta F_1)\big\rangle_1 \tag{2.20}$$

for every $s = 1, 2, \ldots$.

Furthermore, differentiation (2.16) with respect to t gives

$$W_{tt}^s + MW_t^s = F_{1tt}. \tag{2.21}$$

By (2.16),

$$W_t^s|_{t=0} = W_t^{s-1}|_{t=T} + W_0(x) \tag{2.22}$$

where $W_0(x) \equiv M^2 w_T - T\delta(F_{1t} + MF_1)$. From (2.18) we have

$$W_t^s|_{\partial\Omega} = 0. \tag{2.23}$$

Repeating the arguments led to (2.19) and (2.20) we obtain the estimates

$$\int_0^T \|W_t^{s+1} - W_t^s\|_2^2\,dt \le (m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)} \big\langle W_0, MW_0\big\rangle_1, \tag{2.24}$$

$$\int_0^T \|W_{tt}^{s+1} - W_{tt}^s\|^2\,dt \le m_4(m_3^2 - \alpha m_2)^{-1} e^{-\alpha T(s-1)} \big\langle W_0, MW_0\big\rangle_1 \tag{2.25}$$

for every $s = 1, 2, \ldots$.

The inequalities (2.14), (2.15), (2.19), (2.20), (2.24), (2.25) implies that the sequence w^s has a limit $w \in L^2(0,T;W_2^4(\Omega))$ and $w_t^s \to w_t$ in $L^2(0,T;W_2^4(\Omega))$ as $s\to+\infty$. Then $w^s \to w$ in $C([0,T];W_2^4(\Omega))$ as $s\to+\infty$ which implies that there exist the traces $w^s(0,x), w^s(T,x) \in W_2^2(\Omega)$ and $w^s(0,x)\to w(0,x)$, $w^s(T,x)\to w(T,x)$ in $W_2^4(\Omega)$ as $s\to+\infty$. Passing to the limit in (2.5)–(2.6) we conclude that w satisfies equation (2.2) for almost all $(t,x)\in Q_T$, the data (2.3) for almost all $x\in\overline{\Omega}$. By (2.7), the boundary condition (2.4) also asserts for w.

Step 2. Let us come back to the problem (1.7)–(1.12). Using the definition of w and (1.2)–(1.4) we can now find u_1

$$u_1 = w + u_0 - w(0,x) - h + h(0,x) \tag{2.26}$$

and express U_2 from (2.1) in terms of u_1 as

$$U_2 \equiv \int_0^T u_2\, dt = \big[\delta u_1 + M\overline{u}_1 - g_1(x)U_1 - \overline{f}_1\big](g_2(x))^{-1} \equiv \psi(x). \tag{2.27}$$

Thus, we obtain the problem for equation (1.8) on u_2 with the conditions (1.12) and (2.27). By the smoothness of w and (2.26), $u_1 \in C([0,T]; W_2^4(\Omega))$, $\psi(x) \in W_2^2(\Omega)$. According Theorem 1.2 the problem (1.8), (1.12), (2.27) has a unique solution $u_2 \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_2^1(\Omega))$ and $u_{2t} \in L^2(0,T; W_2^{-1}(\Omega))$. The theorem is proved.

Let us consider Problem 1 with the equation

$$u_{1t} + Mu_1 = g_1(x)U_1 + AU_2 + f_1(t,x) \tag{2.28}$$

instead of (1.6) where A is an operator of the form

$$A = -\mathrm{div}(\mathcal{A}(x)\nabla) + (\mathbf{a}, \nabla)_R + a(x)I \tag{2.29}$$

where $\mathcal{A}(x) \equiv (a_{ij}(x))$ is a matrix of functions, $i,j = 1,2,\dots,n$; $\mathbf{a} = \mathbf{a}(x)$ is a vector function; $a(x)$ is a scalar function. If there exist positive constants α_k, $k = 1,2$, such that for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$\alpha_1 \|v\|_1^2 \le \langle Av, v\rangle_1 \le \alpha_2 \|v\|_1^2, \tag{2.30}$$

then the operator A is invertible and Theorem 2.1 remains true for the problem (1.7)–(1.9),(2.28).

Theorem 2.2. *Let the assumptions of Theorem 2.1 be fulfilled. Let $a_{ij}(x) \in W_\infty^1(\Omega)$, $i,j = 1,2,\cdots,n$, $\boldsymbol{a} \in (L^\infty(\Omega))^n$, $a \in L^\infty(\Omega)$ and (2.30) holds. Then the problem (1.7)–(1.9),(2.28) has a unique solution $\{u_1, u_2\} \in V$ and $u_{1tt} \in L^2(0,T; W_2^2(\Omega))$.*

Proof. Let us consider the problem (1.8),(1.9),(2.28). We integrate (2.28) with respect to t on $[0,T]$ and divide the result by T again. In view of (1.8) this yields

$$\delta u_1 + M\bar{u}_1 = g_1(x)U_1 + g_2(x)AU_2 + \bar{f}_1. \tag{2.31}$$

Subtracting (2.31) from (2.28) and rewriting the last equality in terms of the function w we come to the problem (2.2)–(2.4). As was proved, this problem has a unique solution $w \in C([0,T]; W_2^4(\Omega))$, $w_t \in L^2([0,T]; W_2^4(\Omega))$ and $w_{tt} \in L^2(Q_T)$. Using the definition of w we find u_1 and express U_2 from (2.31) in terms of u_1 as

$$U_2 = q(x) + A^{-1}\psi(x) \tag{2.32}$$

where q satisfies the equation $Aq = 0$ and $q|_{\partial\Omega} = \int_0^T \beta_2 dt$. Thus, we obtain the problem for equation (1.8) on u_2 with the conditions (1.12) and (2.32). In the hypotheses of the corollary $q \in W_2^2(\Omega)$, $u_1 \in C([0,T]; W_2^4(\Omega))$and $\psi(x) \in W_2^2(\Omega)$. Hence $q(x) + A^{-1}\psi(x) \in W_2^2(\Omega)$. According to Theorem 1.2 the problem (1.8), (1.12), (2.27) has a unique solution $u_2 \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_2^1(\Omega))$ and $u_{2t} \in L^2(0,T; W_2^{-1}(\Omega))$, which completes the proof.

3. The problem with the nonlocal condition for u_1

Let us now consider Problem 2. We are interested in finding the sufficient conditions for the existence and uniqueness of a solution to Problem 2. By a solution of Problem

2 we mean the pair $\{u_1, u_2\}$ of the class $V_1 = \{\{u_1, u_2\} | u_k \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_2^1(\Omega)),\ u_{kt} \in L^2(0, T; W_2^{-1}(\Omega)), k = 1, 2\}$ which satisfies (1.10)–(1.12).

We suppose that the operators M, L and B satisfy the following conditions instead of I and II.

III. The coefficients $m_{ij}(x), l_{ij}(x), b_{ij}(x) \in W_\infty^1(\Omega)$, $i, j = 1, 2, \cdots, n$; $\mathbf{m}, \mathbf{l}, \mathbf{b} \in (L^\infty(\Omega))^n$, $m, l, b \in L^\infty(\Omega)$.

IV. The operators M, L and B are of elliptic type, that is, M and L obey (1.4), (1.5) and there exist positive constants k_B, K_B, such that for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$k_B \|v\|_1^2 \le \left\langle Bv, v \right\rangle_1 \le K_B \|v\|_1^2. \tag{3.1}$$

Theorem 3.1. *Let the assumptions III, IV be fulfilled and $\partial\Omega \in C^2$. Let also*

iii) $f_1 \in L^2(0, T; W_2^{-1}(\Omega))$, $\mu \in L^2(\Omega)$, $\varphi \in W_2^2(\Omega)$, $\beta_1 \in L^\infty(0, T; W_2^{1/2}(\partial\Omega))$, $\beta_{1t} \in L^2(Q_T)$, $f_2 \in L^2(Q_T)$, $\beta_2 \in L^\infty(0, T; W_2^{1/2}(\partial\Omega))$;

iv) $a(x, p)$ *is continuous in* $\overline{\Omega} \times (-\infty, +\infty)$ *and* $\|a(x, v)\| < +\infty$ *for all* $v \in W_2^1(\Omega)$.

Then Problem 2 has a unique generalized solution $\{u_1, u_2\} \in V_1$.

Proof. We prove the theorem in two steps again. In the first step we find u_2 as a solution of a problem for the equation (1.11) with the boundary data (1.9) and an integral condition with respect to t and establish the uniqueness of u_1. The second step consists of proving the existence and uniqueness of u_1 as the solution of the problem for (1.10) with the boundary data (1.9) and the first condition of (1.12) provided that u_2 is known.

Step 1. Let us integrate (1.10) with respect to t from 0 to T and operate the result with the operator B^{-1} which exists by the assumption II$'$. This gives

$$\int_0^T u_2 dt = B^{-1} \left(\mu(x) + M\varphi(x) - \int_0^T f_1(t, x) dt \right). \tag{3.2}$$

By Theorem 1.2, the problem (1.11),(1.9),(3.2) has a unique solution $u_2 \in Y$ and $u_{2t} \in L^2(0, T; W_2^{-1}(\Omega))$.

Step 2. We now consider the problem for (1.10) with the boundary data (1.9) and the first condition of (1.12). The right term $Bu_2 + f_1$ of (1.10) is known and belongs to $L^2(), T; W_2^{-1}(\Omega))$. By Theorem 1.1, this problem has a unique solution $u_1 \in Y$ and $u_{1t} \in L^2(0, T; W_2^{-1}(\Omega))$.

Thus the solution $\{u_1, u_2\}$ of Problem 2 exists and is unique in the class V_1. Theorem is proved.

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