

ON THE SOLVABILITY OF THE DISCRETE ANALOGUE OF GELFAND — LEVITAN EQUATION

G. B. Bakanov¹, S. I. Kabanikhin^{2,3,4}, N. S. Novikov^{2,4}, M. A. Shishlenin^{2,3,4}

¹*Akhmet Yassawi International Kazakh-Turkish University, Turkestan, Kazakhstan*

²*Institute of Computational Mathematics and Mathematical Geophysics SB RAS, 630090, Novosibirsk*

³*Sobolev Institute of Mathematics SB RAS, 630090, Novosibirsk*

⁴*Novosibirsk State University, 630090, Novosibirsk*

UDC 517.962.8, 519.633.6, 519.642.4

The two-dimensional coefficient inverse problem for the wave equation is considered. We investigate 2D analogue of Gelfand — Levitan equation on the discrete level to reduce the inverse problem to the family of linear integral equations. We consider the solvability conditions for the discrete version of Gelfand — Levitan equation.

Keywords: Coefficient inverse problems, Gelfand — Levitan equation, linear regularization, discrete analogue.

Introduction

Aleekseev and Dobrinskii [3] used the discrete analogy of the Gelfand — Levitan method while investigating the numerical algorithms for a one-dimensional dynamic inverse problem of seismics (see also [1, 2, 5, 8, 9]). The comparison of the Gel'fand-Levitan method with others intended for solving one-dimensional inverse problems can be found in [7]. Pariiskii [4] analyzed the numerical algorithms for solving the Gelfand — Levitan equations. We also should mention another works, that considered numerical solution of 1D and 2D Gelfand — Levitan equations [13, 22, 24, 25]. In Kabanikhin [12, 15] the multi-dimensional analogy of the Gelfand — Levitan equation was obtained. We mention also Berryman and Greene [6], Case and Kack [10], Bube [11], Gladwell and Willms [14], Natterer [16], and a series of more recent works [17–21, 23].

1 Discrete analogue of the Gelfand — Levitan equation

In this paper we consider the following inverse problem:

$$\frac{\partial^2 u^{(m)}}{\partial t^2} = \frac{\partial^2 u^{(m)}}{\partial x^2} + \frac{\partial^2 u^{(m)}}{\partial y^2} - q(x, y)u^{(m)}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad t > 0; \quad (1)$$

$$u^{(m)}(x, y, 0) = 0, \quad \frac{\partial u^{(m)}}{\partial t}(x, y, 0) = \delta(x)e^{imy}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}; \quad (2)$$

$$u^{(m)}(0, y, t) = f^{(m)}(y, t), \quad \frac{\partial u^{(m)}}{\partial x}(0, y, t) = 0, \quad y \in \mathbb{R}, \quad t > 0. \quad (3)$$

We suppose that $q(x, y)$ is even for all variables, and functions $u^{(m)}(x, y, t)$ and $q(x, y)$ are 2π periodical with respect to y . The problem is to recover the unknown function $q(x, y)$ by using the set of measurements data $f^{(m)}(y, t), m \in \mathbb{Z}$.

Such problem arises in geophysics, when the two-dimensional inverse problem for some hyperbolic equation is considered. The idea, used in this paper, is to apply the Gelfand — Levitan method. Gelfand — Levitan — Krein approach is well-studied in one-dimensional case (see introduction), but the two-dimensional case is yet to be fully

The work was supported by RFBR (grants 16-29-15120 and 15-01-09230) and the Ministry of Education and Science of the Republic of Kazakhstan, grant MES 1746/GF4 (project NTP 04.03.02).

investigated.

We will use the discrete version of the approach [17, 18], which fits well into the applied nature of inverse problems (due to the fact, that in practice we have only the finite number of time points, where the data of inverse problems are measured). Let us continue functions $u^{(m)}(x, y, t)$, $f^{(m)}(y, t)$ as odd functions for $t < 0$ and denote $(\tilde{f}_j^k)_t^m = (f_j^k)_t^m - \delta_h^k e^{imjh_1}$, where

$$(f_j^k)_t^m = \begin{cases} 1/h, & k = 0, \\ 0, & k = \pm 1, \\ (f_j^{|k|+1} - f_j^{|k|-1})/2h, & k = \pm 2, \pm 3, \dots \end{cases}$$

We consider then the discrete analogue of inverse problem (1)–(3): find the grid function $\tilde{q}_{i,j}$ from

$$\tilde{v}_{tt}^m = \tilde{v}_{xx}^m + \tilde{v}_{yy}^m + \frac{1}{2} [\delta_{i+k}^h + \delta_{i-k}^h] [(e^{imjh_1})_{y\bar{y}} + e^{imjh_1} \tilde{q}_{i,j}] - \tilde{q}_{i,j} (\tilde{v}_{i,j}^k)^m, \quad i \in \mathbb{Z}, \quad k \geq 1, \quad j, m = \overline{1, 2N_1}; \quad (4)$$

$$(\tilde{v}_{i,j}^0)^m = 0, \quad (\tilde{v}_{i,j}^1)^m = 0, \quad i \in \mathbb{Z}, \quad j, m = \overline{1, 2N_1}; \quad (5)$$

$$(\tilde{v}_{0,j}^0)^m = 0, \quad (\tilde{v}_{0,j}^1)^m = 0, \quad (\tilde{v}_{0,j}^k)^m = (\tilde{f}_j^k)_t^m, \quad k > 1, \quad j, m = \overline{1, 2N_1}. \quad (6)$$

Here over-bar over v corresponds to the finite-difference operators. We will prove that the discrete analogue of inverse problem (1)–(3) is equivalent to the finite set of systems of algebraic equations. Suppose that grid functions $\tilde{q}_{i,j}$, $\tilde{v}_{i,j}^k$ are $2N_1$ -periodical with respect to j . Now we reduce to the two-dimensional discrete inverse problem to the system one-dimensional discrete inverse problems by applying Fourier series expansion with respect to discrete variable j . Let us denote number N_1 , that describes the number of the Fourier coefficients, that we need to restore (that means, that N_1 describes the number of the sources and receivers). Let us introduce matrices \tilde{V} , Q , \tilde{W} — square matrices of $2N_1$ degree, which corresponds to Fourier coefficients of the functions \tilde{v} , \tilde{q} , \tilde{F} — matrix, that consists of $[\tilde{f}_{(n)}^k]_t^m = [f_{(n)}^k]_t^m - \delta_h^k \delta_{m,n}$, O , E — null and unitary matrix correspondingly. Then the problem (4)–(6) can be reduced [17] to the following systems of discrete one dimensional inverse problems:

$$\tilde{W}_i^k + h \sum_{j=-i+1}^{i-1} \tilde{W}_i^j \tilde{F}_t^{k-j} = -\frac{1}{2} [\tilde{F}_t^{k-i} + \tilde{F}_t^{k+i}], \quad 0 \leq |k| < i, \quad i = \overline{2, N}. \quad (7)$$

This discrete version of Gelfand — Levitan equation allows to obtain the solution of inverse problem Q_i by using the function W_i^k , calculated from (7), according to the ratio below:

$$\tilde{W}_{i+1}^i = \frac{1}{2} h \sum_{s=0}^i Q_s, \quad i = 1, 2, \dots \quad (8)$$

The connection between solvability of the discrete inverse problem and the equation (7) is given by the following theorem:

Theorem [17, 18]: Let us suppose that the solution of discrete inverse problem (4)–(6) exists. The for any $i = \overline{2, N}$ system (7) has unique solution. On the other hand, if the system (7) has unique solution \tilde{W}_i^k , then the solution of discrete inverse problem (4)–(6) exists and is unique.

We should also mention several remarks. First, according to the structure of the matrix Q , it is possible to recover $\tilde{q}_{(n)i}$, $n = \overline{0, 2N_1 - 1}$ by using only the first column or the last line of matrix Q_i . Second, we should note, that the parameter $N_1 > 0$, which describes the accuracy of Fourier transform, is arbitrary. Therefore if $f^m(y, t)$ is smooth that we can pass to limit under $N_1 \rightarrow \infty$ and obtain the solution of the initial inverse problem.

2 Solvability conditions

In this section we obtain the solvability condition for the discrete version of the method. As it was mentioned, for every given set of measurements the solvability of leads to solvability of initial inverse problem. Therefore, due to the practical nature of the data of inverse problem, it is important to study the criteria for the existence of Gelfand — Levitan equation's solution (as well as the uniqueness). When one considers the numerical solution of inverse problem both in model case or in case of the real data, one have to work with the finite set of measurements, possessed by measurement's errors, and the criteria mentioned could be used to process data (in the manner, that

would allow us to solve the inverse problem) .

We start from one-dimensional case. Then matrices \tilde{W} , \tilde{F} reduces to numbers, and we can rewrite (7) in the following form:

$$D_i^f w \Leftrightarrow w_i^k + h \sum_{j=-i+1}^{i-1} w_i^j f'^{k-j} = -\frac{1}{2} [f'^{k-i} + f'^{k+i}], 0 \leq |k| < i. \quad (9)$$

Because discrete function f'^k is even function with respect to k , we can consider the discrete Fourier transform of the function f'^k on the interval $t_k \in (-2T, 2T)$:

$$f'^k = \sum_{l=-N}^N a_l \cos \frac{l\pi t_k}{2T}, \quad t_k \in (-2T, 2T). \quad (10)$$

Here

$$a_l = \frac{1}{2T} \sum_{k=-N}^N f'^k \cos \frac{k\pi t_l}{2T}. \quad (11)$$

We also assume, that

$$w_i^k = \sum_{l=-N}^N \varphi_l \cos \frac{\pi l x_k}{ih}, \quad x_k = kh, \quad 0 \leq |k| < i;$$

$$\varphi_l = \frac{1}{ih} \sum_{k=-i}^i w_i^k \cos \frac{\pi k x_l}{ih}.$$

The solvability's condition for the equation (9) can be rewritten in the following form:

$$(D_i^f w, w) > 0 \quad \forall i = 1, \dots, N, \quad \forall w \neq 0. \quad (12)$$

Using the discrete Fourier transform, we rewrite this condition in the following form:

$$\begin{aligned} (D_f^i w, w) &= \frac{1}{ih} \sum_{k=-i}^i D_f^i w_i^k \cdot w_i^k = \frac{1}{ih} \sum_{k=-i}^i (w_i^k)^2 + \frac{1}{(ih)^2} \sum_{j=-i+1}^{i-1} \sum_{k=-i}^i w_i^j w_i^k f'^{k-j} = \\ &= \frac{1}{ih} \sum_{k=-i}^i (w_i^k)^2 + \sum_{j=-i+1}^{i-1} \sum_{k=-i}^i w_i^j w_i^k \sum_{l=-N}^N a_l \cos \frac{l\pi t_{k-j}}{2T} = \\ &= \frac{1}{ih} \sum_{k=-i}^i (w_i^k)^2 + \sum_{l=-i}^i a_l \left[\left(\frac{1}{ih} \sum_{k=-i}^i w_i^k \cos \frac{l\pi t_k}{2T} \right)^2 + \left(\frac{1}{ih} \sum_{k=-i}^i w_i^k \sin \frac{l\pi t_k}{2T} \right)^2 \right]. \end{aligned} \quad (13)$$

Using the (13), we can ensure the inequality (12) either by requiring that

$$a_l > 0 \quad \forall l = -N, \dots, N$$

or

$$1 + 2ih \sum_{l=-N}^N a_l \theta(-a_l) > 0. \quad (14)$$

Another approach, that could be used, relies on the Toeplitz structure of the matrix (9). Let us introduce the generating function Φ — function, which provides diagonals $\{f'^{k-j}\}$ of the matrix as its Fourier coefficients (which corresponds to the inverse Fourier transform):

$$f'^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x) e^{ikx} dx. \quad (15)$$

We should mention, that because of identity matrix in the left hand side of (9), we suppose, that $f'^0 = 1$ in the formula above.

According to Grenander-Szegö theorem [26], values of generating function Φ describes the behaviour of the eigenvalues of the matrix of the system (9). More precisely, the following theorem holds:

Theorem. Let us assume, that the generating function $\Phi(x)$ belongs to $C[-2\pi, 2\pi]$. Denote $\lambda_{\min}(D_N), \lambda_{\max}(D_N)$ — minimal and maximal eigenvalues of the matrix of the system (9), when $i = N$. Then

$$\Phi_{\min} \leq \lambda_{\min}(D_N) \leq \lambda_{\max}(D_N) \leq \Phi_{\max} \quad (16)$$

Moreover, the eigenvalues $\lambda_j(D_N)$ of the system are equally distributed as $\Phi(\frac{2\pi j}{2N})$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{2N-1} \left[h(\lambda_j(D_N)) - h(\Phi(\frac{2\pi j}{2N})) \right] = 0$$

for any $h \in C[-2\pi, 2\pi]$.

Therefore, if

$$\Phi_{\min} > 0, \quad (17)$$

then the matrix of the system (9) is positive definite, which leads to the existence and uniqueness of the solution of (9), and the inverse problem as well. If we have large enough number of data points, then we can calculate the generating function $\Phi(x)$ as the inverse Fourier transform with sufficient accuracy and use the condition (16).

We should mention, that both conditions (14) and (17) can be used together or independently from each other in order to estimate the ability to obtain accurate approximation of inverse problem's solution from the given data. We should also mention, that both results can be modified for the two-dimensional case in the same manner. Accurate modification of considered methods will be published later elsewhere.

References

- [1] Goupillaud, P. (1961). An approach to inverse filtering of nearsurface layer effects from seismic record. *Geophysics* **26**, 754–760.
- [2] Kunetz, G. (1961). Essai d'analyse de traces sismiques. *Geophysical Prospecting* **9**, No. 3, 317–341.
- [3] Alekseev, A. S. and Dobrinskii, V. I. (1975). Questions of practical application of dynamical inverse problems of seismology. In: *Mathematical Problems of Geophysics*, Vol. 6, No. 2. Computer Center, Siberian Branch of USSR Academy Sci., Novosibirsk, pp. 7–53 (in Russian).
- [4] Pariiskii, B. S. (1978). An economical method for the numerical solution of convolution equations. *USSR Computational Math. and Math. Phys.* **17**, No. 2, 208–211.
- [5] Symes, W. W. (1979). Inverse boundary value problems and a theorem of Gel'fand and Levitan. *J. Math. Anal. Appl.* **71**, 378–402.
- [6] Berryman, J. G. and Greene, R. R. (1980). Discrete inverse methods for elastic waves. *Geophys.* Vol. 45, 213–233.
- [7] Burridge, R. (1980). The Gelfand — Levitan, the Marchenko and the Gopinath-Sondhi integral equation of inverse scattering theory, regarded in the context of inverse impulse-response problems. *Wave Motion* **2**, 305–323.
- [8] Symes, W. W. (1981). Stable solution of the inverse reflection problem for a smoothly stratified elastic medium. *SIAM J. Math. Anal.* **12**, No. 3, 421–453.
- [9] Santosa, F. (1982). Numerical scheme for the inversion of acoustical impedance profile based on the Gelfand — Levitan method. *Geophys. J. Roy. Astr. Soc.* **70**, 229–244.
- [10] Case, K. M. and Kack, M. (1983). A discrete version of the inverse scattering problem. *J. Math. Phys.* Vol 4, 594–603.
- [11] Bube, K. (1984). Convergence of discrete inversion solutions. In: *Inverse Problems of Acoustic and Elastic Waves*, SIAM pp. 20–47.

- [12] Kabanikhin, S.I. (1988a). *Linear Regularization of Multidimensional Inverse Problems for Hyperbolic Equations*. Preprint No. 27. Institute Math., Siberian Branch of USSR Academy Sci., Novosibirsk (in Russian).
- [13] Belonosov V.S., Skazka V.V. The inverse dynamic problem of seismic sounding low-frequency regularization. *Applied Mathematics Letters*, 21(1), pp. 95–100, 2008
- [14] Gladwell, G.M.L. and Willms, N.B. (1989). A discrete Gel’fand-Levitan method for band-matrix inverse eigenvalue problems. *Inverse Problems*. Vol. 5, 165–179.
- [15] Kabanikhin, S.I. (1990). On linear regularization of multidimensional inverse problems for hyperbolic equations. *Sov. Math. Dokl.* **40**, No. 3, 579–583.
- [16] Natterer, F. (1994). A discrete Gelfand — Levitan theory. *Technical report, Institut fuer Numerische und instrumentelle Mathematik*. Universitaet Muenster, Muenster, Germany.
- [17] Kabanikhin S.I., Bakanov G.B. A discrete analogue of the gel’fand-levitan method // (1997) *Doklady Mathematics*. T. 56, No 2. C. 670–673.
- [18] Kabanikhin S.I. and Bakanov, G.B. (1999). A discrete analogue of the Gelfand — Levitan method in a two-dimensional inverse problem for a hyperbolic equation // *Siberian Math. J.* **40**, no. 2, 262–280.
- [19] Kabanikhin, S.I., Shishlenin, M.A. Boundary control and Gelfand — Levitan — Krein methods in inverse acoustic problem // (2004) *Journal of Inverse and Ill-Posed Problems*, 12 (2), pp. 125–144.
- [20] Kabanikhin S. I., Satybaev A. D., Shishlenin M. A. Direct methods of solving inverse hyperbolic problems. 2004. VSP/BRILL, the Netherlands.
- [21] Kabanikhin, S.I., Shishlenin, M.A. Numerical algorithm for two-dimensional inverse acoustic problem based on Gelfand — Levitan — Krein equation // (2011) *Journal of Inverse and Ill-Posed Problems*, 18 (9), pp. 979–995.
- [22] Kabanikhin, S.I., Novikov, N.S., Oseledets, I.V., Shishlenin, M.A. Fast Toeplitz linear system inversion for solving two-dimensional acoustic inverse problem // (2015) *Journal of Inverse and Ill-Posed Problems*, 23 (6), pp. 687–700.
- [23] Kabanikhin, S.I., Shishlenin, M.A. Two-dimensional analogs of the equations of Gelfand, Levitan, Krein, and Marchenko // (2015) *Eurasian Journal of Mathematical and Computer Applications*, 3 (2), pp. 70–99.
- [24] Kabanikhin, S.I., Sabelfeld, K.K., Novikov, N.S., Shishlenin, M.A. Numerical solution of the multidimensional Gelfand — Levitan equation // (2015) *Journal of Inverse and Ill-Posed Problems*, 23 (5), pp. 439–450.
- [25] Kabanikhin, S.I., Sabelfeld, K.K., Novikov, N.S., Shishlenin, M.A. Numerical solution of an inverse problem of coefficient recovering for a wave equation by a stochastic projection methods. // (2015) *Monte Carlo Methods and Applications*, 21 (3), pp. 189–203.
- [26] Chan, R. and Jin, X. *An Introduction to Iterative Toeplitz Solvers*, Society for Industrial and Applied Mathematics, 2007

Galitdin Bakanov — Dr. Phys.-Math.Sci.,

Akhmet Yassawi International Kazakh-Turkish University;

e-mail: galitdin.bakanov@ayu.edu.kz;

Sergey Kabanikhin — Dr. Phys.-Math.Sci., director of ICMMG SB RAS;

e-mail: ksi@sscc.ru;

Nikita Novikov — junior research fellow of ICMMG SB RAS;

e-mail: novikov-1989@yandex.ru;

Maxim Shishlenin — Dr. Phys.-Math.Sci., senior research fellow of ICMMG SB RAS;

e-mail: mshishlenin@ngs.ru.

Received — May 30, 2017.