

H. POINCARÉ BALAYAGE METHOD AND NUMERICAL REALIZATION OF THE METHOD ON BASE OF POTENTIAL THEORY

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Several mathematical models for gravity potential on base of time-dependent diffusion equation are constructed in the article. We proposed set of boundary conditions. Numerical realizations H. Poincaré balayage method for solving the boundary problems are constructed. We considered inverse for the balayage method concentration method and the numerical concentration.

Key words: the mathematical models of balayage, swelling, concentration, field potential, simple layer.

Introduction

Dirichlet problem — first kind boundary-value problem for Poisson equation of potential field has important mean for mathematical physics. B. Riemann solution of the problem was disproved by K. Weierstrass (1869 y.). It had great difficulties for mathematicians [4]. H. Poincaré developed an ingenious and general method for Dirichlet problem (1890 y.). He generalized 2D alternating Schwartz method. The method was used for proof of many theorems of mathematical theory (including potential theory). Also principle of the method realized by mesh method. However, the method has not wide using for solving Dirichlet problem. H. Poincaré proposed using for practice Neumann method. V.A. Steklov used Neumann method. Besides, Riemann solution of the problem was proved by D. Hilbert (1900).

At present, Poincaré method is adapted to geophysics on base of D. Zidarov work [3] where was considered inverse gravity problem, inverse magnetic problem and inverse electric problem. There was proposed guideline solving of the inverse problems for complicated geometrical objects (for describing of a deposit) based on the balayage method. Development and application the guideline for the masses has place in the work V.N. Strakhov [8], for oil deposits - in work of V.G. Filatov [9], for S-approximation — in the works of I.E. Stepanovoy and D.N. Rayevsky [5]. Also, principle of balayage in defined measure has place in Yagola method of extending compacts [10] and the works A. Hasanoglu.

1 Balayage, swelling and concentration

Let us now consider Poincaré balayage method. We have a body is filled of a mass and has the potential is specified on boundary of the body. The body is being filled set of intersecting spheres fit closely to boundary of the body. Further we start iterative loop. On first iteration all masses from the spheres are redistributing to the surfaces of the spheres. On next iterations we also have redistributing of the masses from the spheres filling the domain to the spheres's surfaces. Note that with second iteration all inside masses are distributed only on intersection parts of the surfaces of the spheres. This iterative process of H. Poincaré is endless. By the process we search the potential of these masses on the boundary of the domain like the potential is defined in first kind boundary condition. Problem redistributing the masses had not interest for Poincaré.

D. Zidarov's work [3] pay attention to outside for the boundary Γ of the body Ω potential during the balayage process and the characteristics of the body. As noted, the potential field of redistributing masses in each iteration of Poincaré method is coinciding outside of the body, and the potential field is decreasing on the boundary. The

balayage process is executing until all the masses redistributed on the boundary Γ as a simple layer (in numerical realization- with some exactness). The densities of the masses during redistributing are changing. For the method of swelling some spheres with null masses adding to the body after that we are executing iteration process of balayage for the new object. So, if we have not additional conditions for a family of bodies (which we define below), we have infinity set of the bodies are received from the body with the same external (out of the body) potential, but with different (decreasing) potentials on the boundary of the body. During the balayage process the potential inside the body is decreasing.

The family of the bodies is set of the points or the bodies with equally mass are limited by the boundaries Γ_i , $i = 1, \dots, N$ disposed on the distance $h_i(X)$, $X \in \Gamma_i$ and having similar form.

For the families of the bodies we can consider the points with equally masses, the domains with equally masses are limited with the concentric spheres, the domains with equally masses are limited with the confocal ellipsoids. In the general way, the families of the bodies are the domains with equally masses are limited with Liapunov's surfaces [6].

Let us now remember some definitions [3].

The accessible simple layer is called the simple layer (on the closed surface or on the free surface) with points of which we can move to infinity not intersecting this layer in other points.

The mother body for the family is called the body in which before the balayage process is concentrated the mass evoked the anomalous gravitational potential.

The mother layer is called the accessible simple layer corresponding the boundary of the mother body on which all masses of the mother body are sweep.

We shall consider as the mother body one of limiting cases of described above the families of the bodies. For example, we can consider the point mass m as the limiting case of all bodies with defined potential m/R and the mass m . In the same way, we can consider the body is bounded the elliptical surface which is limit of the confocal ellipsoids for which the minor axis converges to zero. This mother body and the described set of the bodies have the same external potentials and the masses.

The inside point of the body T is called the point which not belongs to T and this point can not to move out of the body to infinity not intersecting the points of T .

The frame of the body T is called the set of points including the body and it's inside points.

The frame of the body is sensible for the domain which is limited no less than two surfaces of same dimension. That is we consider the bodies which have the positive distributing density and the negative inside density (the negative density corresponding parts of the domain without the masses).

For the mother body of the family, we may balayge the masses by the spheres only on the mother layer or to outside of the mother body on the accessible simple layers of the family [3]. In the general case, it can be considered the frame of the mother body. Thus we define the mother body among of the family of the bodies equivalent by the external potential by defining the mother layer. In that case the mother layer can be defined as the accessible simple layers with zero points. The zero points X_j^0 , $j = 1, \dots, M$ are the points of the layer with zero density $\rho(X_j^0) = 0$. However, no each family of the bodies with positive densities has unique the mother body. Therefore, it is need conditions of uniqueness for the mother body of the family. These conditions connected with the conditions of uniqueness for inverse gravimetry problem. If the mother body is defined then the iterative balayage process is unique. Example of the mother body for all bodies with potential U_m is point mass m with potential U_m .

Balayage problem is considered of D. Zidarov supposing that the external potential of the mother body $\Omega_M \equiv \Omega_M(\omega, \rho)$ in the points $X \notin \Omega_M$ is given $U_M(X) = \int_{\Omega_M} \frac{\rho(\omega)}{r} d\omega$. For it $\rho(\omega)$ - distribution of the masses in

Ω_M , r - distance between the point $\omega(\xi, \eta, \zeta) \in \Omega_M$ with the density $\rho(\omega)$ and the point $X \equiv (x, y, z) \notin \Omega_M$: $r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$. Therefore, the point $\omega(\xi, \eta, \zeta)$ may be described by the mass $m(\omega)$ if ω is center of the small cube with volume $\Delta\omega \subset \Omega_M$. In this case $m(\omega) = \rho(\omega)\Delta\omega$. Consequently, the summarized mass of the mother body $M = \int_{\Omega_M} m(\omega) d\omega$. The family for the mother body is specified. Also, low distribution

of the density $\rho_1(\omega)$ for the body Ω of the family with the mother body is known. $\Omega \supset \Omega_M$ has mass M : $M = \int_{\Omega} m(\omega) d\omega = \int_{\Omega_M} m(\omega) d\omega$ and such external potential as the mother body. It should be found the domain

Ω . Here, X - Cartesian coordinates. Instead of Cartesian coordinate system we can using polar coordinate system (for 2D case) or spherical coordinate system (for 3D case).

In view of equivalence of redistributions mass for balayage we can set the problem of the simple layer with mass M on the boundary $\Gamma = \partial\Omega$ with given the potential $V(X)|_\Gamma$ determination. Let us consider the system of equipotential surfaces with given values of the potentials $V(X) = c_k$ fitting with the potentials of the mother body in the different points K . So we have the system extending equipotential surfaces: $c_1 > c_2 > \dots > c_k$. The less potential in a point of the mother body, the further lies corresponding to the potential the equipotential surface. So, we have K Dirichlet problems.

The problem concentration consist in transformation the body Ω to the body Ω_M so that by balayage of the masses from Ω_M it may be obtained Ω . Here, Ω and Ω_M have the equally masses and the external potentials. The problem of concentration can be reduced to searching the mother layer Γ_M which is the accessible simple layer. For the simple layer we use Liapunov's surface.

2 Mathematical models of balayage

As model of balayage can be considered Dirichlet problem or Neumann problem for the potential. For the model can be used Laplace equation or equation with source.

Second-order differential equations for partial derivatives (hyperbolic, parabolic, elliptic equations) are invariant ones [2] for isotropic and homogeneous medium. Let us remark here that elliptical equation is being using for closed domain. Hyperbolic and parabolic equations have boundary conditions for other than closed boundary. So, for parabolic equation may be used domain which is opened on one side (for example the layer with opened one side). Therefore, parabolic equation extends the set of studied domains. Also, in sense of analyticity of solution, parabolic equation links to elliptical one.

Let us consider now the model of balayage for the gravity field potential of the domain $\Omega \subset S$ with the mass M and the density $\rho(\omega)$. Here, S may be closed or open domain. For example, the domain is a cube or a lower half-space. The potential may be a function of time $U(X, t)$, where for 2D- $X = (x, z)$, for 3D- $X = (x, y, z)$. The model is given by:

$$\begin{aligned}\Delta U(X, t) &= U_t(X, t), \quad X \in S, \quad t \in (0, T) \\ U(X, 0) &= U_0(X), \quad X \in S \\ \frac{\partial U(X, t)}{\partial n} \Big|_\Gamma &= V(X, t)|_\Gamma\end{aligned}$$

With $U_0(X)$:

$$U_0(X) = \begin{cases} U_0^{in}(X), & X \in \Omega \\ U_0^{ex}(X) = U_0^{ex}(X'), & X, X' \in S \setminus \Omega, X \notin \Omega, \end{cases} \quad (1)$$

where for 2D-case $X' = x$, for 3D-case $X' = (x, y)$.

$$U_0^{in}(X) > U_0^{ex}(X) \quad (2)$$

The realization of the balayage process is to construct set of the extending equipotential layers $V(X, t) = c(t)$ with the speeds $V^{max} - \delta, V^{max} - 2\delta, \dots$ (δ — small positive value). The layers are approaching to the layer for maximum value of the potential $V^{max} = \max_S \frac{\partial U_0(X)}{\partial n}$ so that the finish equipotential layer involves the domain Ω . In the process each the equipotential layer is the simple layer with mass is sweep from the domain Ω by Poincare balayage method. It is satisfying to note that the inside potential of the equipotential layer on each iteration of the balayage process decreases. In the end of each iteration the potential on the equipotential layer is more than the inside potential.

Also, as the model of balayage we can consider the model on base of Dirichlet problem:

$$\begin{aligned}\Delta U(X, t) &= U_t(X, t), \quad X \in S, \quad t \in (0, T) \\ U(X, 0) &= U_0(X), \quad X \in S \\ U(X, t)|_\Gamma &= V_\Gamma(X, t)|_\Gamma\end{aligned}$$

In this case, the process of balayage to be executed on the set of the equipotential layers $U(X, t) = c(t)$ approaching to the layer with greatest mean of the potential inside of the domain S : $U^{max} = \max_S U_0(X)$.

For described above process we can consider the model of balayage with moving boundary $\Gamma(t) : \Gamma_0 = \Gamma(0) \rightarrow \Gamma(T), t \rightarrow T$, where $S = S(T)$ is the domain limited by $\Gamma(T)$. In the general case $\Omega = \bigcup_{i=1}^N \Omega_i$ and correspondingly $\Gamma^0 = \Gamma(0) = \bigcup_{i=1}^N \Gamma_i^0$.

$$\Delta U(X, t) = U_t(X, t), \quad X \in S, \quad t \in (0, T) \quad (3)$$

$$U(X, 0) = U_0(X), \quad X \in S \quad (4)$$

$$\Gamma_0 = \Gamma(0) = \partial\Omega, \quad \frac{\partial\Gamma(X, t)}{\partial t} = -D \frac{\partial U(X, t)}{\partial n} \Big|_{X \in \Gamma(t)}, \quad \Gamma_T = \Gamma(T) = \partial S \quad (5)$$

$$\frac{\partial U(X, t)}{\partial n} \Big|_{\Gamma(t)} = V_\Gamma(X, t) \quad (6)$$

Instead of the condition (6) it can be considered

$$\int_0^T \frac{\partial U(X, t)}{\partial n} \Big|_{\Gamma(t)} dt = V^{sum}(X, T) \quad (7)$$

It is intended that $\Gamma(t)$ involves part of the domain containing the mass. The mass is being sweeping to $\Gamma(t)$. Thus, according to N.M. Gunter researches [1] we may separate the inside for $\Gamma(t)$ mass from the outside mass also the inside for $\Gamma(t)$ potential of the mass from the outside one if there are known normal derivatives of potentials of inside and outside layers on $\Gamma(t)$ -condition (6). This condition is result of the more common condition for separating the inside potential of the mass for $\Gamma(t)$ and the outside one. The common condition consists in knowing the integral normal derivative on all the surfaces $\Gamma(t)$ of the domains with mass-condition (7).

The model with moving boundary we can consider with first kind boundary condition.

$$\Delta U(X, t) = U_t(X, t), \quad X \in S, \quad t \in (0, T) \quad (8)$$

$$U(X, 0) = U_0(X), \quad X \in S \quad (9)$$

$$\Gamma_0 = \Gamma(0) = \partial\Omega, \quad \frac{\partial\Gamma(X, t)}{\partial t} = -D \frac{\partial U(X, t)}{\partial n} \Big|_{X \in \Gamma(t)}, \quad \Gamma_T = \Gamma(T) = \partial S \quad (10)$$

$$U(X, t)|_{\Gamma(t)} = V_\Gamma(X, t) \quad (11)$$

Instead of the condition (11) it can be considered the integral one:

$$\int_0^T U(X, t)|_{\Gamma(t)} dt = U^{sum}(X, T) \quad (12)$$

Any one of (11) (or (12)) or (6) (or (7)) is Gunter condition for separating the inside potential of the mass of searching domain Ω and the outside one.

The mathematical models discussed above can be applied for several gravitating masses of the domain $\Omega = \bigcup_{i=1}^N \Omega_i$.

3 Numerical balayage method

The numerical realization of the mathematical models discussed above is grid realization of H. Poincare balayage method for the masses. Let us cover 3D domain S by the grid with the step h . In the domain Ω_h the point lattice approximation of the sphere is 7 dot scheme type of "cross". So, the grid domain is filling with the point lattice spheres. Further, the iteration balayage process for the mass is starting. Let us consider various realizations of the numerical balayage.

Let we know the density $\rho_\Omega(\omega)$ of the domain Ω (correspondingly Ω_h) and the density $\rho_S(X)$ of the domain S (S_h). The domains Ω and S have equal the outside potential. We seek the domain S (S_h). Then for each

the dot scheme cross in Ω_h using the density in the center of the cross which is more than the preset value. We divide the excess density by 6 and sweeping one to others 6 points of the “cross”. In the center of the “cross” the density $\rho_S(X_h)$ remains. If the density in the center is not excess one then this density not change. This process is performing for all crosses filling the domain Ω_h . Further we execute the next iteration. The iterative process continuing until we have the domain S_h with apriori density of mass distribution. The process of swelling defines transformation Ω to involving the domain by sweeping.

Second realization of the numerical balayage assumes that all mass will sweep to the boundary of the domain S . Therefore, on the first iteration we divide the density in the center of the cross into 6 and redistribute one to others 6 points of the cross. Iteration loop repeats until all the mass is distributed to ∂S . The result layer and the source body Ω have the same mass. The potential of the result layer is equal the outside potential Ω on ∂S . Thus, we construct the equipotential surface including the body Ω .

The iteration loop for the density ρ of the cube S_h on each the iteration consists of 3 nested loops for $i = 2, \dots, I-1$, $j = 2, \dots, J-1$, $k = 2, \dots, K-1$:

$$\begin{aligned}\rho_{i-1,j,k}^n &= \rho_{i-1,j,k} + 1/6\rho_{i,j,k}, \quad \rho_{i+1,j,k}^n = \rho_{i+1,j,k} + 1/6\rho_{i,j,k} \\ \rho_{i,j-1,k}^n &= \rho_{i,j-1,k} + 1/6\rho_{i,j,k}, \quad \rho_{i,j+1,k}^n = \rho_{i,j+1,k} + 1/6\rho_{i,j,k} \\ \rho_{i,j,k-1}^n &= \rho_{i,j,k-1} + 1/6\rho_{i,j,k}, \quad \rho_{i,j,k+1}^n = \rho_{i,j,k+1} + 1/6\rho_{i,j,k}, \quad \rho_{i,j,k}^n = 0 \\ \rho_{i-1,j,k} &= \rho_{i-1,j,k}^n, \quad \rho_{i+1,j,k} = \rho_{i+1,j,k}^n, \quad \rho_{i,j-1,k} = \rho_{i,j-1,k}^n, \quad \rho_{i,j+1,k} = \rho_{i,j+1,k}^n \\ \rho_{i,j,k-1} &= \rho_{i,j,k-1}^n, \quad \rho_{i,j,k+1} = \rho_{i,j,k+1}^n, \quad \rho_{i,j,k} = \rho_{i,j,k}^n\end{aligned}$$

The truncation condition of the iteration loop is the discrepancy between initial and final (boundary) masses less than the accuracy ϵ .

4 Method of concentration

The problem of concentration uses the inverse procedure for the balayage problem. Let us consider various realizations of the concentration. In the process of the density concentration we move from the layer on the external surface to the layer on the equipotential surface with greater potential value (the internal surface). This process has N iterations. So, we have set of N the equipotential layers: $V = c$, $V = c + \Delta V$, \dots , $V = c + (N-1)\Delta V$. For various models we may use the set of the simple layers or the double ones. On each surface in the points with the normal derivative of the potential discontinuity it is left the part of the mass (consequently, the part of the potential). We may leave the N -th part of all mass M of first external surface.

The density continuation with the boundary performing if the conditions: $\rho(X) > 0$ and $\frac{\partial \rho(X)}{\partial n} < 0$ are true. If one of the conditions is not true then we leave part of the mass on the boundary (on the surface). So, for each step k if the one of the conditions is not true we leave $\rho_k(X) = \rho_k^\Gamma(X) + \rho_k^\Phi(X)$ on k -th boundary.

Let us consider some numerical realizations of the concentration. In the first realization we know distribution of the density in the enveloping body and distribution of one in the resolving body. The process using the iteration loop where for each template of cross type we move the masses. Thus, 6-th equal masses m in the nodes of the cross must be replaced with zero and the mass $6m$ must be relocated to center of the cross. If we have dissimilar masses in the cross nodes then $6m_{min}$ (m_{min} — minimal mass of the cross nodes) must be relocated to center of the cross. The masses $m_i - m_{min}$ must be remained to the cross nodes $i = 1, \dots, 6$. The procedure must be executed for all crosses in the mesh domain S_h . For concentration with the boundary the sum of the masses of 6-th cross nodes must be relocated to center of the cross. So for anyone mesh approximation of the sphere. Finally, the algorithm of sweeping from the inside domain S may be used for obtaining the apriori potential on the boundary ∂S .

5 Results

The balayage algorithm and the concentration one are experienced on the set of the model examples in the cube $S = \{\xi, \eta, \zeta\}$, $\xi \in [0, 1]$, $\eta \in [0, 1]$, $\zeta \in [1/3, 4/3]$. We considered the set of the point bodies: 1, 2, 4, 8, 16 points with the density $1g/cm^3$ from the oil density segment. The set of sweepings determines hits on the levels point masses. Maximum effect of the balayage for inside point observed in three nearest points of the boundary ∂S .

The results of the model experiments is presented with tables of values. On the base of the tables constructed the balayage densities maps. The calculation time is a few seconds.

The concentration for the indicated points are executing by algorithm of iterative balayage with using the maps of the balayage densities, method Monte-Carlo for the densities and statistical regularization. Using of the maps lets reduce the calculating time. By using information about bottom and top means of the oil density $[\rho_{oil}^{bttm}, \rho_{oil}^{top}]$ we exclude non-perspective sources. In our case, the optimal iteration number of Monte-Carlo method is 300. For accurate input data and distance the average error of concentration is less than 1.5%. For input data and discrepancy the average error is fewer than 5%. The more the number of sources the more the computation time. Nevertheless for CPU frequency $1GHz$ calculating time for 16 sources is equal to 1 hour 40 minutes.

The methods are used for searching set of the oil deposits.

Conclusion

In the article the models of balayage for the potential of gravity field and the numerical method on base of Poincare balayage method are considered. This approach is convenient in solving inverse gravity, magnetic and electric problems by the method of concentration.

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References

- [1] Gunter N.M. Potential theory and application to main problems of mathematical physics. M.: GITTL, 1953.
- [2] Zommerfeld A. Differential equations in partial derivatives of physics. Moscow: IL. 1950.
- [3] Zidarov D. About some inverse problems of potential fields and their application to questions of geophysics. Sofia: BAN. 1964.
- [4] H. Poincare. Selected works. Volume 3. Moscow: Nauka, 1974.
- [5] Raevsky D.N. Development the new effective itarational methods of soluting the systems algebraic equations of inverse geophysical problems on base of integral representation method. Author's abstract. Moscow: IFZ. 2016.
- [6] Steklov V.A. Fundamental problems of mathematical physics. Moscow: Nauka. 1983.
- [7] Sveshnikov A.G., Bogolubov A.N., Kravtsov V.V. Lectures on mathematical physics. M.: Nauka. 2004.
- [8] Strakhov V.N. About balayage of masses Poincare and its using for solution gravimetry dirrect and inverse problems //DAN USSR. 1977. Volume 236, N 1, P. 54–57.
- [9] Filatov V.G. Stable methods interpretation of potential fields on base of regularization and concentration of the sources. Author's abstract. Kiev. 1988.
- [10] Yagola A.G., Dorofeev K.Y. Sourcewise representation and a posteriori error estimates for ill-posed problems //Vestnik Moscow University. Series 3. Physics, astronomy. Moscow: Moscow University Press. N 2. P. 64–66. 1999.
- [11] Yagola A.G., Wang Yanfei, Stepanova I.E., Titarenko V.N. Inverse problems and the methods of solution. Geophysics applications. Moscow: BINOM. Knowledge laboratory. 2014.

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