

DIVERSE BLOW-UP REGIMES IN NONLINEAR DIFFUSION PROCESSES

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Physical phenomena with critical blow-up regimes simulated by the 3D nonlinear diffusion equation in a spherical shell are studied. For solving the model numerically, the original differential operator is split along the radial coordinate, as well as an original technique of using two coordinate maps for solving the 2D subproblem on the sphere is involved. This results in 1D finite difference subproblems with simple periodic boundary conditions in the latitudinal and longitudinal directions that lead to unconditionally stable implicit second-order finite difference schemes. A band structure of the resulting matrices allows applying fast direct (non-iterative) linear solvers using the Sherman-Morrison formula and Thomas algorithm. The developed method is tested in several numerical experiments. Our tests demonstrate that the model allows simulating different regimes of blow-up in a 3D complex domain. In particular, heat localisation is shown to lead to the breakup of the medium into individual fragments followed by the formation and development of self-organising patterns, which may have promising applications in thermonuclear fusion, nonlinear inelastic deformation and fracture of loaded solids and media and other areas.

Keywords: Evolution with blow-up, nonlinear reaction-diffusion, finite difference schemes, operator splitting.

Introduction

The phenomenon of diffusion has many manifestations in nature. One of the most obvious is from gas dynamics and atmospheric environment. The air, as it is known, is a mixture of different gases, such as carbon dioxide, oxygen, hydrogen, nitrogen and particles of dust. However, due to diffusion the atmospheric composition at any given altitude is fairly homogeneous. The driving force for diffusion is the difference between the thermodynamic potentials. By means of the redistribution of the substance a system tends to equalise local differences of the potentials, and, consequently, is approximated to the thermodynamic equilibrium, and this alignment is carried out by diffusion.

For certain diffusion (heat transfer) models, the charge and momentum, taken in blow-up regimes, are the quantities able to model the formation and evolution of tornadoes and lightnings. Specifically, in [7] it is shown that in the life cycle of tornadoes experiencing in blow-up modes the tornado elephant trunk descends from the atmosphere to the ground. For atmospheric flows of regional and global scales simulation in spherical geometry is appropriate, which leads to 2D (on the sphere) or 3D (in a spherical shell) nonlinear diffusion models. More information on nonlinear diffusion phenomena and applications can be found, e.g., in [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 16, 17, 18].

The development of numerical methods for solving the diffusion equation on a sphere (2D) or in a spherical shell (3D) has its own features. The point is that the sphere is not a doubly periodic domain, since it is periodic in the longitude, but is not in the latitude due to the presence of two poles. Therefore, a numerical procedure designed to solve the original 2D problem will be computationally cumbersome, because the matrix of the resulting linear system will be of a general type, hence not permitting to apply some or other fast linear solvers. As for making some or other modifications to the model prior to computing, these normally bring to the necessity of constructing mathematically and physically correct boundary conditions or involving special numerical procedures near the poles (e.g., matrix bordering). Both are always a problem, as the poles represent an artificial boundary appearing exclusively due to the latitudinal-longitudinal coordinate system, and the construction of proper artificial boundary conditions is a serious independent question.

Some of the three-dimensional nonlinear diffusion equations considered in the spherical geometry are convenient to solve by separating the diffusion operator into the spherical and radial components. The main idea consists in splitting the original differential operator by coordinates and subsequent constructing finite difference schemes for the split 1D subproblems using two different coordinate maps for the same sphere when computing in the longitudinal and latitudinal directions. The key advantage of this technique is that each split 1D equation can be equipped with a periodic boundary condition, despite the sphere being not a doubly periodic domain. Therefore, unlike the existing methods, this one does not require applying special numerical procedures for careful computing the solution near the poles, which is always a serious challenge. The developed algorithm is cheap-to-implement from the computational point of view [12, 13, 14, 15]. The theoretical results are confirmed numerically by simulating various nonlinear diffusion processes.

1 Nonlinear Diffusion Model

We consider the 3D nonlinear diffusion equation in the spherical shell written in the coordinates r (the radius), λ (the longitude), φ (the latitude)

$$\frac{\partial T}{\partial t} = (A_r + A_\lambda + A_\varphi)T + f, \quad (1)$$

where

$$A_r T = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 D \frac{\partial T}{\partial r} \right), \quad A_\lambda T = \frac{1}{r \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{D}{r \cos \varphi} \frac{\partial T}{\partial \lambda} \right), \quad A_\varphi T = \frac{1}{r \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{D \cos \varphi}{r} \frac{\partial T}{\partial \varphi} \right). \quad (2)$$

Here $D \sim (T)^\alpha$ is the diffusion coefficient, while f is the external forcing. Using the standard method of operator splitting, we define the grid spacing in time as $\tau = t_{n+1} - t_n$ and in every double time interval (t_{n-1}, t_{n+1}) linearise (1) and then split the linearised equation by coordinates, assuming $D \sim (T^{n-1})^\alpha$ and $T^{n-1} = T(\lambda, \varphi, t_{n-1})$. The split equations are to be solved in time successively: the solution to the radial subproblem (i.e. solved in r) is the initial condition for the longitudinal subproblem; that, in turn, after being solved in λ , will provide the initial condition for the latitudinal subproblem; the latter, solved in φ , will supply the initial condition for the forcing equation. Next, in order to have the approximation order $O(\tau^2)$, the bicyclic splitting procedure is used [8] — the order of the splitting changes to the inverse one (see formulas (5)-(11) below).

Because the metric term $r \cos \varphi$ vanishes at $\varphi = \pm \frac{\pi}{2}$, the spherical operators A_λ and A_φ are meaningless at the poles. Therefore, for approximating the entire sphere S , under $\Delta \lambda = \lambda_{k+1} - \lambda_k$ and $\Delta \varphi = \varphi_{l+1} - \varphi_l$ we define on S a grid making a half step shift in φ as

$$S_{\Delta \lambda, \Delta \varphi}^{(1)} = \left\{ (\lambda_k, \varphi_l) : \lambda_k \in \left[\frac{\Delta \lambda}{2}, 2\pi + \frac{\Delta \lambda}{2} \right), \varphi_l \in \left[-\frac{\pi}{2} + \frac{\Delta \varphi}{2}, \frac{\pi}{2} - \frac{\Delta \varphi}{2} \right] \right\}. \quad (3)$$

Due to the shift in φ we exclude the pole singularities, so that the resulting finite difference equations will have sense everywhere on $S^{(1)}$. For subsequent computing in the longitude one may, evidently, use the periodic boundary condition, as the sphere S is a periodic domain in λ . As for computing in the latitude, the point is that the sphere is not a periodic domain in φ . Several approaches can be used to solve the latitudinal subproblem, e.g. one could involve a matrix bordering procedure or try to paste the solution from the opposite meridians at the poles. A disadvantage of these techniques is that in the case of an implicit temporal approximation the resulting matrix will be of a general type and so no fast algorithms of linear algebra can be used for computing the solution. Alternatively, an explicit temporal discretisation can be used, but this will impose serious restrictions on the time step, especially when α is large. However, this is exactly the method of splitting that allows to avoid these undesired procedures which, if performed inaccurately, may easily result in introducing nonphysical modes into the solution. Due to the splitting, for computing the solution in φ we change the coordinate map from (3) to

$$S_{\Delta \lambda, \Delta \varphi}^{(2)} = \left\{ (\lambda_k, \varphi_l) : \lambda_k \in \left[\frac{\Delta \lambda}{2}, \pi - \frac{\Delta \lambda}{2} \right], \varphi_l \in \left[-\frac{\pi}{2} + \frac{\Delta \varphi}{2}, \frac{3\pi}{2} + \frac{\Delta \varphi}{2} \right] \right\}. \quad (4)$$

Obviously, grid (4) contains the same nodes as (3). The use of the two coordinate maps $S^{(1)}$ and $S^{(2)}$ allows employing the numerical algorithm with the periodic boundary conditions in both directions, λ and φ [13, 14, 15]. The only change we have to make in A_φ if using (4) is to replace $\cos \varphi$ with $|\cos \varphi|$.

We take the second-order finite difference approximations

$$\begin{aligned}\frac{\partial}{\partial r} \left(r^2 D \frac{\partial T}{\partial r} \right) &\approx \frac{1}{\Delta r} \left((r^2 D)_{p+\frac{1}{2}} \frac{T_{p+1} - T_p}{\Delta r} - (r^2 D)_{p-\frac{1}{2}} \frac{T_p - T_{p-1}}{\Delta r} \right), \\ \frac{\partial}{\partial \lambda} \left(D \frac{\partial T}{\partial \lambda} \right) &\approx \frac{1}{\Delta \lambda} \left(D_{k+\frac{1}{2}} \frac{T_{k+1} - T_k}{\Delta \lambda} - D_{k-\frac{1}{2}} \frac{T_k - T_{k-1}}{\Delta \lambda} \right), \\ \frac{\partial}{\partial \varphi} \left(D |\cos \varphi| \frac{\partial T}{\partial \varphi} \right) &\approx \frac{1}{\Delta \varphi} \left((D |\cos \varphi|)_{l+\frac{1}{2}} \frac{T_{l+1} - T_l}{\Delta \varphi} - (D |\cos \varphi|)_{l-\frac{1}{2}} \frac{T_l - T_{l-1}}{\Delta \varphi} \right).\end{aligned}$$

Supplementing them with the bicyclic splitting [8]

$$\frac{T_{pkl}^{n-\frac{3}{4}} - T_{pkl}^{n-1}}{\tau} = A_r \frac{T_{pkl}^{n-\frac{3}{4}} + T_{pkl}^{n-1}}{2}, \quad (5)$$

$$\frac{T_{pkl}^{n-\frac{2}{4}} - T_{pkl}^{n-\frac{3}{4}}}{\tau} = A_\lambda \frac{T_{pkl}^{n-\frac{2}{4}} + T_{pkl}^{n-\frac{3}{4}}}{2}, \quad (6)$$

$$\frac{T_{pkl}^{n-\frac{1}{4}} - T_{pkl}^{n-\frac{2}{4}}}{\tau} = A_\varphi \frac{T_{pkl}^{n-\frac{1}{4}} + T_{pkl}^{n-\frac{2}{4}}}{2}, \quad (7)$$

$$\frac{T_{pkl}^{n+\frac{1}{4}} - T_{pkl}^{n-\frac{1}{4}}}{2\tau} = f_{pkl}^n, \quad (8)$$

$$\frac{T_{pkl}^{n+\frac{2}{4}} - T_{pkl}^{n+\frac{1}{4}}}{\tau} = A_\varphi \frac{T_{pkl}^{n+\frac{2}{4}} + T_{pkl}^{n+\frac{1}{4}}}{2}, \quad (9)$$

$$\frac{T_{pkl}^{n+\frac{3}{4}} - T_{pkl}^{n+\frac{2}{4}}}{\tau} = A_\lambda \frac{T_{pkl}^{n+\frac{3}{4}} + T_{pkl}^{n+\frac{2}{4}}}{2}, \quad (10)$$

$$\frac{T_{pkl}^{n+1} - T_{pkl}^{n+\frac{3}{4}}}{\tau} = A_r \frac{T_{pkl}^{n+1} + T_{pkl}^{n+\frac{3}{4}}}{2}, \quad (11)$$

we achieve the second order accuracy in time in the interval (t_{n-1}, t_{n+1}) , provided $D \sim (T^{(n-1)})^\alpha$.

The solution T^{n+1} to (5)-(11) converges to the solution to the unsplit 3D linearised differential problem in the interval (t_{n-1}, t_{n+1}) under $\tau \rightarrow 0$.

2 Numerical Tests

In Figure 1 we simulate a nonlinear diffusion phenomenon called temperature waves. Here $f \sim T - T^3$, which results in the following solution: while the area of the temperature is expanding, the maximum value of T does not change, i.e. the wave front is of constant value.

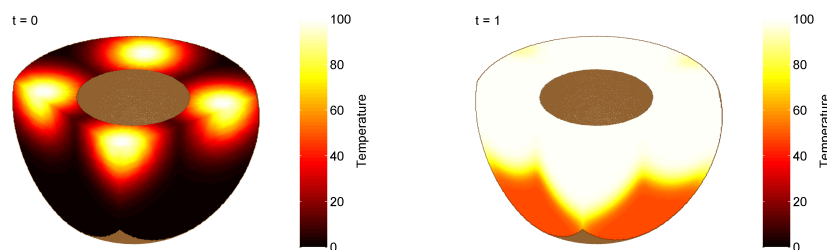


Figure 1: The initial condition ($t = 0$) and the final solution ($t = 1$) of equation (1) with $f \sim T - T^3$. This simulates the phenomenon of temperature waves with a constant front

In Figure 2 we provide the initial and final solutions of equation (1) for a more sophisticated case — the phenomenon of combustion [11]. There are three different modes of combustion depending on the parameter β

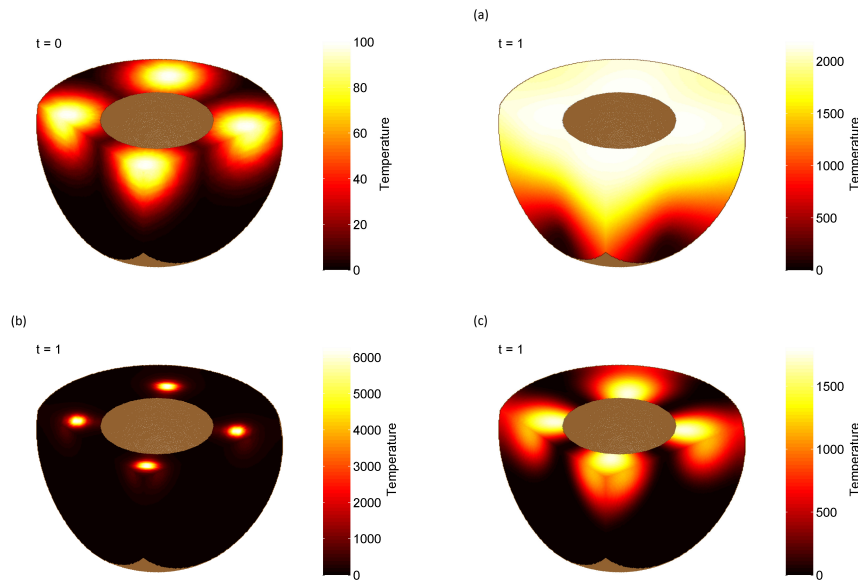


Figure 2: The initial condition ($t = 0$) and the final solution ($t = 1$) of equation (1) with $f \sim T^\beta$, where $\beta < \alpha + 1$ (the area of combustion is expanding, (a)), $\beta > \alpha + 1$ (the area of combustion is reducing, (b)) and $\beta = \alpha + 1$ (the area of combustion is constant, (c))

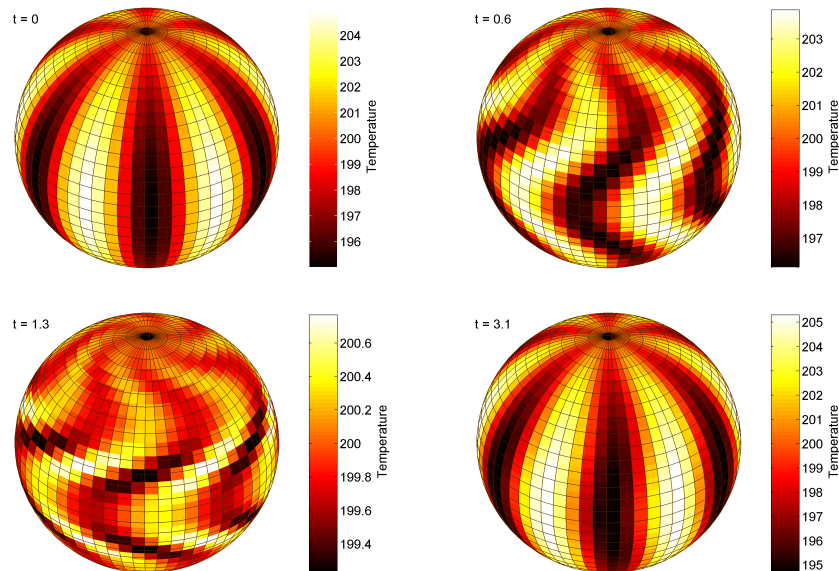


Figure 3: The initial condition ($t = 0$) and several solution of equation (1) for the spherical travelling waves phenomenon. This is an example of nonlinear interaction between diverse mechanisms of the model — external forcing and dissipation, which results in solutions with periodic breakups and renewals of complex structures via self-organisation

that introduces a nonlinearity into the sources $f \sim T^\beta$: the case $\beta < \alpha + 1$ corresponds to the so-called HS-regime — the area of combustion is getting expanded while burning; the case $\beta > \alpha + 1$ corresponds to the LS-regime — the area of combustion is getting narrower; the case $\beta = \alpha + 1$ corresponds to the constant area of burning (S-regime). In all the cases the nonlinear feedback due to the sources leads to an infinite increase of the

temperature T over time, that is a blow-up occurs [11].

In Figure 3 we show a yet another phenomenon — the spherical travelling waves. These are modelled by the diffusion equation with a specially chosen right-hand side function f . Specifically, the sinusoid begins travelling clockwise at the low latitudes and near the poles, while at the intermediate latitudes it goes anticlockwise. After some time the process alters, and the rotation in the opposite direction takes place (not shown). This phenomenon evinces the interaction between the nonlinearity, external forcing and dissipation, which generates solutions with periodic breakups and renewals of complex structures via the self-organisation process.

Conclusion

We have developed an efficient non-iterative implicit numerical method for the solution of the 3D nonlinear diffusion problem with a nonlinear forcing that generates unbounded solutions. Our method is based on the coordinate splitting and yields unconditionally stable finite differences schemes of second approximation order in all the directions. Three practically important blow-up regimes with an infinite growth of the solution over a finite time were successfully simulated: in the HS-regime the solution grows in an expanding domain, in the LS-regime it grows in a shrinking domain, and in the S-regime the growth takes place in a fixed-size area. Our method can be used in various applications, e.g. for simulating extremely growing solutions in rapid compression and accumulation of matter (laser fusion), chemical kinetics and magnetohydrodynamics, meteorology (tornadoes and lightnings), ecology (growth and extinction of biological populations), neurophysiology and epidemiology (infectious disease outbreaks), economics (rapid economic growth), demography (world population growth) — just to name a few.

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