

GLOBAL SEARCH FOR OPTIMIZATION PROBLEM WITH D.C. GOAL FUNCTION AND D.C. CONSTRAINTS

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This paper addresses the general nonconvex problem with the goal function and constraints given by d.c. functions. We reduce this problem to a problem without constraints by the exact penalty approach. Relations between the original and the penalized problems are investigated. In addition, employing the d.c. structure of penalized problem the Global Optimality Conditions (GOCs) are developed and analyzed. We prove that the GOCs possess the constructive property. Moreover, it is shown that the point satisfying the GOCs turns out to be a KKT-vector in the original problem. Besides we establish that the verification of the GOCs consists in a solution of a family of the partially linearized problems, and consecutive verification of the principal inequality of the GOCs. The effectiveness of the GOCs is verified by a number of examples in which the GOCs confirm its ability to escape stationary points and local minima with improving the goal function.

Keywords: nonconvex optimization, d.c. functions, exact penalty, global optimality conditions.

Introduction

A little more than 50 years ago (!!!) the exact penalty method was invented by I.I. Eremin and W.I. Zangvill [1–3] ([2], p. 272–287, see Notes in p. 285) independently and almost simultaneously.

Since that time, this approach has been widely used and nowadays becomes increasingly popular. It is viewed by specialists as a very powerful and effective tool for solving difficult real-life problems, including mathematical models of conflict situations (games, equilibria, bilevel problems, hierarchical control etc.) [2, 3, 8, 11, 13–17].

On the other hand, almost all real-life optimization problems are explicitly or implicitly nonconvex, with a lot (often a huge number!) of stationary vectors and local pits [4–6].

For such problems the classical optimization methods provide only the KKT points [1–9]. On the other hand, the methods of Global Optimization (B&B, cut’s method etc.) [4–6] suffer the so-called “curse of dimension”, when the exponential growth of computational efforts corresponds to an increase in dimension of the problem in question [4–6].

In this paper we continue to develop the apparatus of the so-called Global Optimality Conditions (GOC) [25–30] for the nonconvex optimization problems, the goal function and inequality constraints of which are given by d.c. functions (the difference of convex functions).

This time we use popular techniques of the Exact Penalization [1–3, 10–17] to reduce the original problem to a nonconvex (penalized) problem without inequality constraints. For this purpose, we apply the well-known existence results for the threshold exact penalty parameter obtained by F. Clarke and J. Burke [11, 13]. In addition, one can use the results from [1–3], which develop the corresponding theorems of S. Han and O. Mangasarian [14], as well as very interesting and original papers by professor G di Pillo et al. [15–17]. It is worth nothing that, like all specialists in optimization, we approve a considerable influence of the new mathematical apparatus created in the last two decades [10–13, 18–24] and in the end of the 20th century.

The current research develops the new GOCs [25–30] for the problem in question with the help of the exact penalty approach and reduces the original problem to a problem without inequality constraints (Section 2). After that, we represent the cost function of the penalized problem as a d.c. function. The latter helps us develop the necessary GOCs, which turn out to be related to the KKT Theorem in the original problem. In Section 3,

we identify and explain various features of the GOC developed, in particular, the constructive property. Besides, we use illustrative examples to demonstrate that the GOCs successfully escape local pits. Finally, we give the sufficient GOCs in Theorem 3.

The next section presents the statement of the original problem.

1 Statement of the problem

Consider the following problem:

$$(\mathcal{P}): \left. \begin{aligned} f_0(x) &:= g_0(x) - h_0(x) \downarrow \min_x, \quad x \in S, \\ f_i(x) &:= g_i(x) - h_i(x) \leq 0, \quad i \in I = \{1, \dots, m\} \\ f_j(x) &:= g_j(x) - h_j(x) = 0, \quad j \in \mathcal{E} = \{m+1, \dots, l\}; \end{aligned} \right\}$$

where the functions $g_i(\cdot)$, $h_i(\cdot)$, $i \in I \cup \mathcal{E} \cup \{0\}$, are convex on \mathbb{R}^n , so that the functions $f_i(\cdot)$, $i \in I \cup \mathcal{E} \cup \{0\}$, are the d.c. functions [4–6, 10, 21]. Henceforth, assume that $g_i(\cdot)$ and $h_i(\cdot)$ are differentiable on an open set Ω , and S is a convex set $S \subset \Omega \subset \mathbb{R}^n$.

Besides, suppose that the set $Sol(\mathcal{P})$ of global solutions to Problem (\mathcal{P}) , $Sol(\mathcal{P}) = \{x \in \mathcal{F} \mid f_0(x) = \mathcal{V}(\mathcal{P})\}$, and the feasible set \mathcal{F} of Problem (\mathcal{P}) , $\mathcal{F} := \{x \in S \mid f_i(x) \leq 0, i \in I, f_j(x) = 0, j \in \mathcal{E}\}$, are non-empty. In what follows the optimal value $\mathcal{V}(\mathcal{P})$ of Problem (\mathcal{P}) is supposed to be finite:

$$\mathcal{V}(\mathcal{P}) := \inf(f_0, \mathcal{F}) := \inf_x \{f_0(x) \mid x \in S, f_i(x) \leq 0, i \in I, f_j(x) = 0, j \in \mathcal{E}\} > -\infty.$$

We have to recall that, since $g_i(\cdot)$, $h_i(\cdot)$, $i \in I \cup \mathcal{E} \cup \{0\}$, are convex functions, they are locally Lipschitz [10, 11, 18, 19, 22]. Therefore, $f_i(\cdot)$, $i \in I \cup \{0\}$, are also locally Lipschitz functions [10, 11, 18–22].

2 Exact penalization

Introduce the penalty function $W(\cdot)$ for Problem (\mathcal{P}) as follows

$$W(x) := \max\{0, f_1(x), \dots, f_m(x)\} + \sum_{j \in \mathcal{E}} |f_j(x)| = \|F_{\mathcal{I}}(x)\|_{\infty} + \|F_{\mathcal{E}}(x)\|_1, \quad (1)$$

where $F_{\mathcal{I}}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $F_{\mathcal{E}}: \mathbb{R}^n \rightarrow \mathbb{R}^{l-m}$ are defined as $F_{\mathcal{I}}(x) := (f_1^+(x), \dots, f_m^+(x))^{\top}$, $f_i^+(x) := \max\{0, f_i(x)\}$, $i \in I$, and $F_{\mathcal{E}}(x) := (f_{m+1}(x), \dots, f_l(x))^{\top}$.

Furthermore, along with Problem (\mathcal{P}) , consider the penalized problem without the constraints:

$$(\mathcal{P}_{\sigma}): \quad \theta_{\sigma}(x) := f_0(x) + \sigma W(x) \downarrow \min_x, \quad x \in S, \quad (2)$$

where $\sigma \geq 0$ is a penalty parameter and the penalized function.

As well-known, if $z \in Sol(\mathcal{P}_{\sigma})$, and $z \in \mathcal{F}$, then $z \in Sol(\mathcal{P})$ [1–3, 10, 11, 13]. However, the inverse implication does not, in general, hold so that the principal moment of the exact penalization (EP) theory is the existence of a threshold value $\sigma_* \geq 0$ of the EP parameter $\sigma \geq 0$ such that $Sol(\mathcal{P}_{\sigma}) \subset Sol(\mathcal{P}) \quad \forall \sigma \geq \sigma_*$.

In other words, for $\sigma \geq \sigma_*$ Problems (\mathcal{P}) and (\mathcal{P}_{σ}) turn out to be equivalent in the sense that $Sol(\mathcal{P}) = Sol(\mathcal{P}_{\sigma})$ (see [10, Chapt. VII, Lemma 1.2.1]). Besides, the term “exact penalty” allows us to solve a single unconstrained problem instead of a sequence of unconstrained problems with $\sigma_k \rightarrow \infty$.

Hence, the proof of existence of the exact penalty threshold σ_* is an important element in the investigation of relations between Problems (\mathcal{P}) and (\mathcal{P}_{σ}) .

Definition 2.1. [1–3, 12–17, 23, 24] The constraint system $\mathcal{F} = \{x \in S \mid f_i(x) \leq 0, i \in I, f_j(x) = 0, j \in \mathcal{E}\}$ satisfies the local error bound property at the point $z \in \mathcal{F}$, if there exist numbers $\varkappa > 0$ and $\rho > 0$ such that

$$dist(x, \mathcal{F}) \leq \varkappa \left(\max\{0, f_1(x), \dots, f_m(x)\} + \sum_{j \in \mathcal{E}} |f_j(x)| \right) \quad \forall x \in S \cap B(z, \rho),$$

where $B(z, \rho) = \{x \in \mathbb{R}^n \mid \|x - z\| \leq \rho\}$.

As well-known, under some constraint qualification (CQ) conditions (RMFCQ, etc. [12, 13, 23, 24]), the local error bound property holds [1–3, 12–14, 23, 24].

On the other hand, G. di Pillo et al. [15–17] developed another approach to prove the existence of a threshold value $\sigma_* \geq 0$ of the exact penalty parameter for the case of a global solution.

Here we follow the way opened by F. Clarke [11] (Proposition 2.4.3) and J.V. Burke [13] (Corollary 2.3.1), aiming to prove the existence of the threshold value $\sigma_* \geq 0$ for a global solution as well. We also have to mention the result from [10] (Chapt. VII, Lemma 1.2.1), and of course the results from [1–3, 12, 14–17, 23, 24] and the references therein.

We say that Problem (\mathcal{P}) satisfies the global error bound property if there exists a number $\varkappa > 0$ such that

$$\text{dist}(x, \mathcal{F}) \leq \varkappa \left(\max\{0, f_1(x), \dots, f_m(x)\} + \sum_{j \in \mathcal{E}} |f_j(x)| \right) \quad \forall x \in S \setminus \mathcal{F}. \quad (3)$$

Henceforth we assume that in Problem (\mathcal{P}) the following assumption holds
 (\mathcal{H}) : the global error bound property takes place.

Proposition 2.1. *Let $z \in \mathcal{F}$ be a global solution to Problem (\mathcal{P}) , $z \in \text{Sol}(\mathcal{P})$, where the function $f_0(\cdot)$ satisfies the Lipschitz property with $L_0 > 0$.*

(i) *Then, there exists $\sigma_* \geq 0$ such that z is a global solution to Problem (\mathcal{P}_{σ_*}) .*

(ii) *Moreover, $\forall \sigma > \sigma_*$ any solution z_σ to Problem (\mathcal{P}_σ) must belong to \mathcal{F} , $z_\sigma \in \text{Sol}(\mathcal{P}_\sigma) \subset \mathcal{F}$, and, therefore, z_σ is a solution to (\mathcal{P}) , so that $\text{Sol}(\mathcal{P}_\sigma) \subset \text{Sol}(\mathcal{P})$.*

(iii) *The later inclusion amounts to the equality*

$$\text{Sol}(\mathcal{P}) = \text{Sol}(\mathcal{P}_\sigma) \quad \forall \sigma \geq \sigma_*, \quad (4)$$

so that Problems (\mathcal{P}) and (\mathcal{P}_σ) turn out to be equivalent (in the sense of (4)).

In what follows, Proposition 2.1 or the similar results from [1–3, 14–17] lay the theoretical basis for further investigation.

3 Optimality conditions

First of all, let us show that the goal function $\theta_\sigma(\cdot)$ of the penalized Problem (\mathcal{P}_σ) is a d.c. function. Indeed, it can be readily seen that

$$\theta_\sigma(x) \triangleq f_0(x) + \sigma \left(\max\{0, f_1(x), \dots, f_m(x)\} + \sum_{j \in \mathcal{E}} |f_j(x)| \right) = G_\sigma(x) - H_\sigma(x), \quad (5)$$

where

$$H_\sigma(x) := h_0(x) + \sigma \sum_{i \in \mathcal{I}} h_i(x) + \sigma \sum_{j \in \mathcal{E}} [g_j(x) + h_j(x)], \quad (6)$$

$$G_\sigma(x) := g_0(x) + \sigma \max \left\{ \sum_{j \in \mathcal{I}} h_j(x); g_i(x) + \sum_{j \in \mathcal{I}}^{j \neq i} h_j(x), \quad i \in \mathcal{I} \right\} + 2\sigma \sum_{j \in \mathcal{E}} \max\{g_j(x), h_j(x)\}. \quad (7)$$

Obviously, $G_\sigma(\cdot)$ and $H_\sigma(\cdot)$ are both convex functions [10, 18, 19], so that the function $\theta_\sigma(\cdot)$ is a d.c. function, as claimed. Furthermore, note that for a feasible (in (\mathcal{P})) point $z \in S$ we have $W(z) \triangleq \max\{0, f_1(z), \dots, f_m(z)\} + \sum_{j \in \mathcal{E}} |f_j(z)| = 0$. Now denote $\zeta := f_0(z)$, so that

$$\theta_\sigma(z) = f_0(z) + \sigma W(z) = f_0(z) = \zeta. \quad (8)$$

Theorem 3.1. *Let a point $z \in \mathcal{F}$ be a solution to Problem (\mathcal{P}) and $\sigma \geq \sigma_* > 0$. Then, for every pair $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ such that*

$$H_\sigma(y) = \beta - \zeta, \quad (9)$$

the following inequality holds

$$G_\sigma(x) - \beta \geq \langle \nabla H_\sigma(y), x - y \rangle \quad \forall x \in S. \quad (10)$$

Remark 3.1. It can be readily seen that Theorem 3.1 reduces the investigation of the nonconvex Problem (\mathcal{P}_σ) to considering a family of the convex linearized problems

$$(\mathcal{P}_\sigma L(y)): \quad \Phi_{\sigma y}(x) := G_\sigma(x) - \langle \nabla H_\sigma(y), x \rangle \downarrow \min_x, \quad x \in S, \quad (11)$$

depending on the parameters $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ which fulfill (9).

It is worth pointing out that the linearization is performed with respect to the “unified” nonconvexity of Problem (\mathcal{P}) expressed by the function $H_\sigma(\cdot)$ (see (\mathcal{P}) –(1), (6)) that accumulates the functions $h_i(\cdot)$, $i \in \{0\} \cup I$, responsible for generating all nonconvexities in (\mathcal{P}) .

On the other hand, the principal inequality (10) can be rewritten, for example, as follows

$$\mathcal{V}(\mathcal{P}L(y)) \geq \beta - \langle \nabla H(y), y \rangle =: \mathcal{N}(y, \beta), \quad (10')$$

where $\mathcal{V}(\mathcal{P}_\sigma L(y))$ is the optimal value of the linearized problem $(\mathcal{P}_\sigma L(y))$.

Remark 3.2. Let there be found a triple (y, β, u) , $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$, $H_\sigma(y) = \beta - \zeta$, $u \in S$ such that the “principal” inequality (10) is violated, i.e.

$$0 > G_\sigma(u) - \beta - \langle \nabla H_\sigma(y), u - y \rangle.$$

Then, with the help of (9) and due to convexity of $H_\sigma(\cdot)$, we derive

$$0 > G_\sigma(u) - \beta - H_\sigma(u) + H_\sigma(y) = \theta_\sigma(u) - \zeta = \theta_\sigma(u) - \theta_\sigma(z),$$

or, $\theta_\sigma(z) > \theta_\sigma(u)$, $z \in \mathcal{F}$, $u \in S$. Hence, the point z can not be a solution to (\mathcal{P}_σ) .

Moreover, according to Proposition 2.1, the point z can not be a solution to (\mathcal{P}) , if $\sigma \geq \sigma_* = \kappa L_0$, because in this case $\text{Sol}(\mathcal{P}) = \text{Sol}(\mathcal{P}_\sigma)$.

Hence, the conditions (9)–(10) of Theorem 3.1 possess the constructive (algorithmic) property (once the conditions are violated, there is a feasible vector which is better than the point in question).

Now, let us turn our attention to the issue of relations and connections, if there are any, between the conditions (9)–(10) of Theorem 3.1 and the Classical Optimization Theory, or, more precisely, the Classical Optimality Conditions. For this purpose, assume that a feasible (in Problem (\mathcal{P})) point z satisfies the conditions (9)–(10).

First, set in (9)–(10) $y = z$. Then it immediately follows that $\beta = H_\sigma(z) + \zeta = G_\sigma(z)$. Furthermore, from (10) we derive that

$$G_\sigma(x) - G_\sigma(z) \geq \langle \nabla H_\sigma(z), x - z \rangle \quad \forall x \in S.$$

It implies that the point z satisfying (9)–(10) is a solution to the linearized problem as follows

$$(\mathcal{P}_\sigma L(z)): \quad G_\sigma(x) - \langle \nabla H_\sigma(z), x \rangle \downarrow \min_x, \quad x \in S. \quad (12)$$

Since $(\mathcal{P}_\sigma L(z))$ is a convex problem, then the following condition is, according to the convex optimization theory, the necessary and sufficient optimality condition for $z \in \text{Sol}(\mathcal{P}_\sigma L(z))$:

$$0_n \in \partial G_\sigma(z) - \nabla H_\sigma(z) + N(z \mid S), \quad (13)$$

where $\partial G_\sigma(z)$ is the classical (convex) subdifferential of the convex function $G_\sigma(\cdot)$ [2, 10, 18, 19] at z , and $N(z \mid S) = \{v \in \mathbb{R}^n \mid \langle v, x - z \rangle \leq 0 \quad \forall x \in S\}$ is the cone of normal vectors to S at $z \in S$. Thus, conditions (9)–(10) entail the classical optimality condition (13) [1–4] for Problem (\mathcal{P}) .

Notwithstanding the partially elucidated relations between the classical Optimization theory (the KKT theorem) and conditions (9)–(10) of Theorem 3.1, there arises a natural question whether it is possible to find a vector $(y, \beta) \in \mathbb{R}^{n+1}$ that satisfies (9) and violates the inequality (10) with some $u \in S$.

The answer is given by the result below

Theorem 3.2. Let a feasible in Problem (\mathcal{P}) point z , $\zeta := f_0(z)$, be given, and, besides, the following condition be fulfilled

$$(\mathcal{H}): \quad \exists v \in \mathbb{R}^n : f_0(v) > f_0(z) \quad (14)$$

In addition, let the penalty parameter $\sigma > 0$ be given, and, besides, suppose that the point z is not a solution to Problem (\mathcal{P}) .

Then, one can find a tuple $(y, \beta, u) \in \mathbb{R}^{2n+1}$, $u \in \mathcal{F}$, such that

$$\left. \begin{array}{l} (a) \quad H_\sigma(y) = \beta - \zeta; \\ (b) \quad G_\sigma(y) \leq \beta, \\ (c) \quad G_\sigma(u) - \beta < \langle \nabla H_\sigma(y), u - y \rangle. \end{array} \right\} \quad (15)$$

Now we present the sufficiency of the Global Optimality Conditions (9)–(10) of Theorem 3.1 using the proof of Theorem 3.2.

Theorem 3.3. *Suppose that for a feasible (in Problem (P)) point z , $\zeta := f_0(z)$, the condition (H)–(14) is fulfilled. Additionally, let for some $\sigma > 0$ and any pair $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$, satisfying the conditions*

$$(a) \quad H_\sigma(y) = \beta - \zeta, \quad (b) \quad G_\sigma(y) \leq \beta, \quad (16)$$

the following inequality hold

$$G_\sigma(x) - \beta \geq \langle \nabla H_\sigma(y), x - y \rangle \quad \forall x \in S. \quad (17)$$

Then, the point $z \in \mathcal{F}$ is a global solution to Problem (\mathcal{P}_σ) as well as to Problem (P).

Example 3.1. Consider the problem ([1, Example 12.20])

$$\left. \begin{aligned} f_0(x) &= 4x_1x_2 \downarrow \min, \quad x \in \mathbb{R}^2, \\ f_1(x) &= x_1^2 + x_2^2 - 1 = 0. \end{aligned} \right\} \quad (18)$$

It is easy to see that the point $z = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^\top$, $\zeta := f_0(z) = 2$, satisfies the KKT-equation

$$\nabla f_0(z) + \lambda_1 \nabla f_1(z) = 0 \in \mathbb{R}^2$$

with $\lambda_1 = -2$. Besides, z is feasible: $f_1(z) = 0$.

However, it is not clear whether the point z is a global solution. In order to decide on it, let us apply Theorem 3.1.

Since $f_0(x) = 4x_1x_2 = (x_1 + x_2)^2 - (x_1 - x_2)^2$, it can be readily seen that

$$\left. \begin{aligned} g_0(x) &= (x_1 + x_2)^2, \quad h_0(x) = (x_1 - x_2)^2, \\ g_1(x) &= x_2^2 + x_2^2, \quad h_1(x) \equiv 1. \end{aligned} \right\} \quad (19)$$

Besides, let set $\sigma := 3 > |\lambda_1| = 2$. Then, according to (6) and (7), we have

$$\left. \begin{aligned} H_\sigma(x) &= h_0(x) + \sigma[g_1(x) + h_1(x)] = (x_1 - x_2)^2 + 3(x_1^2 + x_2^2 + 1), \\ G_\sigma(x) &= g_0(x) + 2\sigma \max\{g_1(x); h_1(x)\} = (x_1 + x_2)^2 + 6 \max\{x_1^2 + x_2^2; 1\}. \end{aligned} \right\} \quad (20)$$

Let choose, now, $y = (-1, 0.5)^\top$ which is unfeasible in the problem (18). Then we have

$$H_\sigma(y) = (y_1 - y_2)^2 + 3(y_1^2 + y_2^2 + 1) = (-1.5)^2 + 3(1 + 0.25 + 1) = 9$$

and, as a consequence, we derive

$$\beta = H_\sigma(y) + \zeta = 9 + 2 = 11.$$

Furthermore, let choose a feasible point $u = (-0.6; 0.8)^\top$, $u_1^2 + u_2^2 = 0.36 + 0.64 = 1$, and compute $G_\sigma(u)$ (see (20))

$$G_\sigma(u) = (u_1 + u_2)^2 + 6 \max\{u_1^2 + u_2^2; 1\} = (0.2)^2 + 6 = 6.04.$$

Besides, it is not difficult to compute that

$$u - y = (-0.6; 0.8)^\top - (-1; 0.5)^\top = (0.4; 0.3)^\top,$$

$$\nabla H_\sigma(y) = 2(y_1 - y_2; y_2 - y_1)^\top + 6(y_1, y_2)^\top = 2(4y_1 - y_2; 4y_2 - y_1)^\top = (-9; 6)^\top.$$

Whence we immediately derive

$$\langle \nabla H_\sigma(y), u - y \rangle = \langle (-9; 6)^\top, (0.4; 0.3)^\top \rangle = -3.6 + 1.8 = -1.8,$$

$$\beta + \langle \nabla H_\sigma(y), u - y \rangle = 11 - 1.8 = 9.2 > 6.04 = G_\sigma(u).$$

The latter inequality means that the principal inequality (10) of Theorem 3.1 is violated.

Hence, the point $z = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^\top$ is not a global solution to the problem (18) in virtue of Theorem 3.1.

Indeed, it is confirmed by the inequality

$$f_0(u) = -0.48 < \zeta = f_0(z) = 2.$$

Conclusion

We consider the general nonconvex problem with the goal function and constraints given by d.c. functions. We reduce this problem to a problem without constraints by the exact penalty approach. Relations between the original and the penalized problems are investigated. In addition, employing the d.c. structure of penalized problem the Global Optimality Conditions (GOCs) [25, 28, 29] are developed and analyzed. We prove that the GOCs possess the constructive property, i.e. when the GOCs are violated, it is possible to find a feasible (in original problem) vector which is better than the point under investigation. Moreover, it is shown that the point satisfying the GOCs turns out to be a KKT vector in the original problem. Besides we establish that the verification of the GOCs consists in a solution of a family of the partially linearized problems, and consecutive verification of the principal inequality of the GOCs. The effectiveness of the GOCs is verified by a number of examples in which the GOCs confirm its ability to escape stationary points and local minima with improving the goal function.

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