Reconstruction of the parameters of time-fractional one-phase Stefan problem

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The 1D time-fractional one-phase Stefan problem is posed in enthalpy form:

$$\begin{gathered} \mathcal{D}_t^\alpha \xi = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad \xi \in H(u), \quad 0 < x < X,\ t > 0, \\ \xi(x,0) = 0, \quad u(0,t) = u_0,\ \ u(X,t) = 0, \end{gathered} \tag{1}$$

where u denotes the temperature,

$$H(u) = \{0 \ \text{ for } \ u < 0; \ \ [0,L] \ \text{ for } \ u = 0; cu + L \ \text{ for } \ u > 0\} \text{ is the enthalpy,}$$

$$\mathcal{D}_t^\alpha \xi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial \xi}{\partial s}(s)ds, \ \ \alpha \in (0,1), \quad \text{is the Caputo derivative.}$$

The stated nonlinear problem with a moving phase transition boundary separating the liquid ($u > 0$) and solid ($u = 0$) phases is approximated by backward Euler scheme on a uniform $h \times \tau$ mesh:

$$\partial_t^\alpha \xi^k + Au^k = f, \quad \xi^k \in H(u^k) \text{ for } k \geq 1, \quad \xi^0 = 0, \tag{2}$$

where A is the finite-difference Laplacian, and the fractional derivative is approximated using the conventional $L1$-method:

$$\mathcal{D}_t^\alpha \xi(t_k) \approx \partial_t^\alpha \xi^k = \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{\xi^j - \xi^{j-1}}{\tau}\int\limits_{s=t_{j-1}}^{t_j}(t_k - s)^{-\alpha}\,ds.$$

The unique solvability of the formulated direct grid problem and the properties of its solution necessary for further analysis are proven.

Two inverse problems are posed as follows:
Given the known position of the point $x^* = s(t^*)$ of the phase transition boundary, restore:

- the order of the fractional derivative α, or
- the constant L of the phase transition heat.

The correctness of the formulation of these inverse problems is justified by the monotonic dependence of the mesh interphase boundary on each of these variable parameters. Inverse problems of identification of the same parameters are posed and investigated for the case of observation of several points of the interphase boundary.

Iterative algorithms for solving the posed grid inverse problems are proposed and an analysis of their effectiveness is carried out.

Several similar inverse problems in a two-dimensional setting have also been solved numerically.

References

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