(M, K)-METHODS FOR CONTROL THEORY PROBLEMS

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PROBLEM DEFINITION

Let us consider the following system of differential-algebraic equations (DAE)

$$\boldsymbol{x' = f(x, y, t),\ x(t_0) = x_0,\ t_0 \le t \le t_k,} \qquad \textbf{(1)}$$

$$0 = \boldsymbol{g(x, y, t).} \qquad \textbf{(2)}$$

Here $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{f}$ and $\boldsymbol{g}$ are sufficiently smooth real-valued vector-functions, $\boldsymbol{t}$ is an independent variable.

Since system **(1), (2)** can be easily reduced to autonomous form by adding equation $\boldsymbol{t'} = 1$, for the sake of simplicity further we assume that **(1), (2)** does not depend on $\boldsymbol{t}$ explicitly.

THE SIGNIFICANCE OF SOLVING DAE

Many control theory problems lead to the necessity of solving **DAE** systems **[1-2]**. For example, these problems arise in

- trajectory prescribed path control;
- chemical reaction management;
- electrical network operation;
- robotics;
- other applications.

[1] Takuma Uchiyama, Hidetsugu Terada and Hironori Mitsuya, Continuous path control of a 5-DOF parallel-serial hybrid robot // Journal of Mechanical Science and Technology Vol. 24. – 2010. – pp. 47-50.

[2] C. C. Pantelides, The consistent initialization of differential-algebraic systems // SIAM J. Sci. Statist. Comput., **9**. – 1988. – pp. 213-231.

THE COMPLEXITY OF ARISING PROBLEMS

Usually on solving control theory problems it is necessary to apply those algorithms, which provide the satisfaction of algebraic constraints **(2)** as accurately as it is possible **[3], [4]**. For this purpose, **usually iterative procedures are used** in calculations. Below general **non-iterative** numerical methods which can be applied **directly to DAE are given**.

Note that **the problems** (optimization of chemical processes, electrical network operation, robotics, etc) mentioned before **are often stiff**. Thus even insignificant perturbations of the parameters of the problem to be solved may lead to great or even inadmissible changes of a solution. **This increases requirements to integration algorithms.**

[3] McGrath M., Howard D., Baker R., The strengths and weaknesses of inverted pendulum models of human walking // Gait \& Posture 41, 2015, p. 389-394

[4] Yao Cai, Qiang Zhan, Xi Xi, Path tracking control of a spherical mobile robot // Mechanism and Machine Theory 51, 2012, p. 58-73

Below the term 'index' as applied to systems of DAE is used. It is defined as follows **[5]**.

System **(1), (2)** is of

a) index 1, if g_x is nonsingular. System **(1), (2)** has an unique solution in this case.

b) index 2, if g does not depend on y and

$$\| g_x f_y \| \le c < \infty .$$

Note that in the latter case the existence and the uniqueness of a solution are not guaranteed **[6]**.

[5] Gear, C. W. Differential-Algebraic Equations Index Transformations // SIAM J. Sci. Stat. Comput, V. 9, No. 1. – 1988. – pp. 39-47.

[6] Rheinboldt, W.C. Differential-Algebraic Systems as Differential Equations on Manifolds // Math. Comput. **43**. – 1984. – pp. 473-482.

ε-EMBEDDING METHOD

Applying the well-known ε-embedding method **[7]**, it is possible to rewrite system **(1), (2)** in the following form:

$$x' = f(x,y),\ x(t_0) = x_0,\ t_0 \le t \le t_k,$$

$$\varepsilon \cdot y' = g(x,y),\ \varepsilon \to 0.$$

OR

$$x' = f(x,y),\ x(t_0) = x_0,\ t_0 \le t \le t_k, \qquad \textbf{(3)}$$

$$y' = g(x,y)/\varepsilon,\ \varepsilon \to 0. \qquad \textbf{(4)}$$

[7] Hairer, E., Wanner, G. Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems. – Springer-Verlag, Berlin. – 1996. – p. 614.

(M, K)-SCHEMES FOR EXPLICIT SYSTEMS

Consider the Cauchy problem for a stiff system:

$$x' = f(x),\ x(t_0) = x_0,\ t_0 \le t \le t_k. \qquad \textbf{(5)}$$

Let m and k be given integers, $m \ge k$. Define sets

$$M_m = \{1, 2, \dots, m\},\ M_k = \{m_i \in M_m \mid m_1 = 1, m_{i-1} \le m_i, 2 \le i \le k, m_k \le m\}.$$

Set M_k contains the stages at which new values of a function are calculated. We introduce sets $J_i = \{j \in M_m \mid j+1 \in M_k, j < i\}$, $1 \le i \le m$, and $\bar{M}_k = M_m \setminus M_k$, which is the addition set of M_k.

Using the introduced notation, **an (*m*, *k*)-scheme [8]** applied to **(5)** **is given on the following slide**.

[8] Novikov E.A., Shornikov Yu.V., Computer simulation of hybrid stiff systems. Novosibirsk: Publishing House of the Novosibirsk State Technical University. – 2012. – p. 451 (in Russian)

(M, K)-SCHEMES FOR EXPLICIT SYSTEMS

$$x_{n+1} = x_n + \sum_{i=1}^{m} \mu_i b_i , \qquad (6)$$

$$D_n b_i = hf\left(x_n + \sum_{j=1}^{i-1} \beta_{ij} b_j\right) + \theta \sum_{j \in J_i} \alpha_{ij} b_j +$$

$$+(1-\theta) hA \sum_{j \in J_i} \gamma_{ij} b_j , \; i \in M_k ,$$

$$D_n b_i = \theta b_{i-1} + \theta \sum_{j \in J_i} \alpha_{ij} b_j + (1-\theta) hA b_{i-1} +$$

$$+(1-\theta) hA \sum_{j \in J_i} \gamma_{ij} b_j , \; i \in \bar{M}_k ,$$

$$D_n = E - ahA .$$

(M, K)-SCHEMES FOR EXPLICIT SYSTEMS

Above μ_i, α_{ij}, β_{ij}, γ_{ij}, θ and a are parameters defining accuracy and stability of scheme **(6)**; h is the integration stepsize; A is a matrix approximating derivative $f_x = \partial f / \partial x$; E is the identity matrix; b_i are stages (increments) of the scheme.

Parameter θ equals either 0 or 1. **On $\theta = 0$ it is more suitable to study accuracy** of **(6)**, whereas **on $\theta = 1$ scheme (6) is more efficient** due to less matrix-vector multiplications.

Applying the well-known ε-embedding method **(m, k)-schemes can be used for solving the systems of DAE as well**.

If the system to be solved is of **index 2**, then an **(m, k)-scheme has the form given on the following slide.**

(M, K)-SCHEMES FOR DAE

$$x_{n+1} = x_n + \sum_{i=1}^{m} \mu_i k_i^x, \; y_{n+1} = y_n + \sum_{i=1}^{m} \mu_i k_i^y, \tag{7}$$

$$D_n \begin{bmatrix} k_i^x \\ k_i^y \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \; D_n = \begin{bmatrix} E - ahf_x & -ahf_y \\ -ahg_x & -ahg_y \end{bmatrix}, \tag{8}$$

$$F_1 = h\delta_i f\left(x_n + \sum_{l=1}^{i-1} \beta_{il} k_l^x, y_n + \sum_{l=1}^{i-1} \beta_{il} k_l^y\right) + \theta \sum_{j \in J_i} \alpha_{ij} k_j^x + (1-\theta) h \sum_{j \in J_i} \gamma_{ij} \left(f_x k_j^x + f_y k_j^y\right),$$

$$F_2 = h\delta_i g\left(x_n + \sum_{l=1}^{i-1} \beta_{il} k_l^x, y_n + \sum_{l=1}^{i-1} \beta_{il} k_l^y\right) + (1-\theta) h \sum_{j \in J_i} \gamma_{ij} \left(g_x k_j^x + g_y k_j^y\right).$$

Here parameter $\delta_i = 1$ for $i \in M_k$ and $\delta_i = 0$ for $i \in \bar{M}_k$.

(3, 2)-SCHEME FOR DAE

$$x_{n+1} = x_n + \sum_{i=1}^{3} \mu_i k_i^x, \; y_{n+1} = y_n + \sum_{i=1}^{3} \mu_i k_i^y, \qquad \textbf{(9)}$$

$$D_n \begin{bmatrix} k_1^x \\ k_1^y \end{bmatrix} = \begin{bmatrix} hf(x_n, y_n) \\ hg(x_n, y_n) \end{bmatrix},$$

$$D_n \begin{bmatrix} k_2^x \\ k_2^y \end{bmatrix} = \begin{bmatrix} hf(x_n + \beta_{21} k_1^x, y_n + \beta_{21} k_1^y) + \alpha_{21} k_1^x \\ hg(x_n + \beta_{21} k_1^x, y_n + \beta_{21} k_1^y) \end{bmatrix},$$

$$D_n \begin{bmatrix} k_3^x \\ k_3^y \end{bmatrix} = \begin{bmatrix} k_2^x + \alpha_{31} k_1^x \\ 0 \end{bmatrix}.$$

Matrix D_n is evaluated by the second formula in **(8)**.

SECOND ORDER CONDITIONS

Parameters of an ***L*-stable** (3, 2)-method providing the **second order** were derived by comparing expansions of exact and approximate solutions in Taylor series. These expansions were obtained for DAE of index 2 using methodology based on the graphical representation of elementary differentials **[9]** and are not given here due to their cumbersomeness. **Below the order conditions are given.**

1. $\mu_1 + \mu_2 = 1,$

2. $\mu_2(\gamma_{21} + \beta_{21}) + \mu_3(\gamma_{31} + \beta_{31}) + \mu_3 = 0.5 - \alpha,$

3. $(\mu_2\alpha - \mu_3)\beta_{21}^2 = \alpha^2,$ **(10)**

4. $(\mu_2\alpha - 2\mu_3)\beta_{21}^2 = 2\alpha^3,$

5. $\mu_3(\gamma_{21} + \beta_{21}) = \alpha^3 - 2\cdot\alpha^2 + 0.5\cdot\alpha,$

[9] Roche, M. Rosenbrock methods for differential algebraic equations // Numer. Math., 1988, V.52, pp. 45-63.

PARAMETERS

To get the parameters of the scheme it is enough to resolve system **(10)**. They are of the form

1. $\mu_1 = (\beta_{21}^2 + 2\alpha^2 - 2\alpha)/\beta_{21}^2$,

2. $\mu_2 = -(2\alpha^2 - 2\alpha)/\beta_{21}^2$,

3. $\mu_3 = (-2\alpha^3 + \alpha^2)/\beta_{21}^2$, **(11)**

4. $\gamma_{21} = (-2\alpha^2 + 4\alpha - 1)\beta_{21}^2/(4\alpha^2 - 2\alpha) - \beta_{21}$,

5. $\gamma_{31} = (4\alpha^3 - 8\alpha^2 + 6\alpha - 1)\beta_{21}^2/(8\alpha^4 - 8\alpha^3 + 2\alpha^2) - 1$.

Formulas connecting the parameters of scheme **(9)** on $\theta = 0$ and $\theta = 1$ are of the form

$$\alpha_{21} = \gamma_{21}/\alpha,\ \alpha_{31} = \gamma_{31} - \gamma_{21}/\alpha,$$

$$p_1 = \mu_1 - \gamma_{21}\mu_2/\alpha + (\gamma_{21} - \gamma_{31})\mu_3/\alpha,\ p_2 = \mu_2 - \mu_3/\alpha,\ p_3 = \mu_3.$$

Let $\theta = 1$ and free parameters α and β_{21} be equal to 1. Then, the method takes the following form...

(3, 2)-METHOD

$$x_{n+1} = x_n + k_1^x + k_2^x - k_3^x,\ y_{n+1} = y_n + k_1^y + k_2^y - k_3^y, \qquad \textbf{(12)}$$

$$D_n \begin{bmatrix} k_1^x \\ k_1^y \end{bmatrix} = \begin{bmatrix} hf(x_n, y_n) \\ hg(x_n, y_n) \end{bmatrix},$$

$$D_n \begin{bmatrix} k_2^x \\ k_2^y \end{bmatrix} = \begin{bmatrix} hf(x_n + k_1^x, y_n + k_1^y) - 0.5 \cdot k_1^x \\ hg(x_n + k_1^x, y_n + k_1^y) \end{bmatrix},$$

$$D_n \begin{bmatrix} k_3^x \\ k_3^y \end{bmatrix} = \begin{bmatrix} k_2^x \\ 0 \end{bmatrix}, \quad D_n = \begin{bmatrix} E - hf_x & -hf_y \\ -hg_x & -hg_y \end{bmatrix}.$$

CHEMICAL AKZO NOBEL PROBLEM

The **first test problem** is given by the stiff system of **6 nonlinear DAE of index 1 [10]**:

$$\begin{cases} x' = f(x, y), \\ 0 = g(x, y), \end{cases} \qquad \textbf{(13)}$$

$$x(0) = x_0, \; y(0) = y_{0,1}, \; 0 \le t \le 180.$$

Functions f and g equal, respectively,

$$f = \begin{pmatrix} -2r_1 + r_2 - r_3 - r_4 \\ -0.5 \cdot r_1 - r_4 - 0.5 \cdot r_5 + F_{in} \\ r_1 - r_2 + r_3 \\ -r_2 + r_3 - 2r_4 \\ r_2 - r_3 + r_5 \end{pmatrix} \quad \text{and} \quad g = K_s \cdot x_1 \cdot x_4 - y_1.$$

[10] F. Mazzia, F. Iavernaro. Test Set for Initial Value Problem Solvers // University of Bari: Department of Mathematics, 2003. 295 p.

CHEMICAL AKZO NOBEL PROBLEM

The auxiliary variables equal

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{k}_1 \cdot \boldsymbol{x}_1^4 \cdot \sqrt{\boldsymbol{x}_2} \\ \boldsymbol{k}_2 \cdot \boldsymbol{x}_3 \cdot \boldsymbol{x}_4 \\ \boldsymbol{k}_2 \cdot \boldsymbol{x}_1 \cdot \boldsymbol{x}_5 \,/\, \boldsymbol{K} \\ \boldsymbol{k}_3 \cdot \boldsymbol{x}_1 \cdot \boldsymbol{x}_4^2 \\ \boldsymbol{k}_4 \cdot \boldsymbol{y}_1^2 \cdot \sqrt{\boldsymbol{x}_2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{F_{in}} = \boldsymbol{klA} \cdot \left(\frac{\boldsymbol{p}(\boldsymbol{CO}_2)}{\boldsymbol{H}} - \boldsymbol{x}_2 \right). \tag{14}$$

Calculations were performed with the following parameters:

$$\boldsymbol{k}_1 = 18.7,\ \boldsymbol{k}_2 = 0.58,\ \boldsymbol{k}_3 = 0.09,\ \boldsymbol{k}_4 = 0.42,\ \boldsymbol{K} = 34.4,\ \boldsymbol{klA} = 3.3,$$

$$\boldsymbol{K_s} = 115.83,\ \boldsymbol{p}(\boldsymbol{CO}_2) = 0.9,\ \boldsymbol{H} = 737.$$

Initial values were: $\boldsymbol{x}_0 = (0.444, 0.00123, 0, 0.007, 0)^{\boldsymbol{T}}$, $\boldsymbol{y}_{0,1} = \boldsymbol{K_s} \cdot \boldsymbol{x}_{0,1} \cdot \boldsymbol{x}_{0,4}$, $\boldsymbol{x}_0' = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0)$ and $\boldsymbol{y}_{0,1}' = \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_{0,1})$.

CHEMICAL AKZO NOBEL PROBLEM

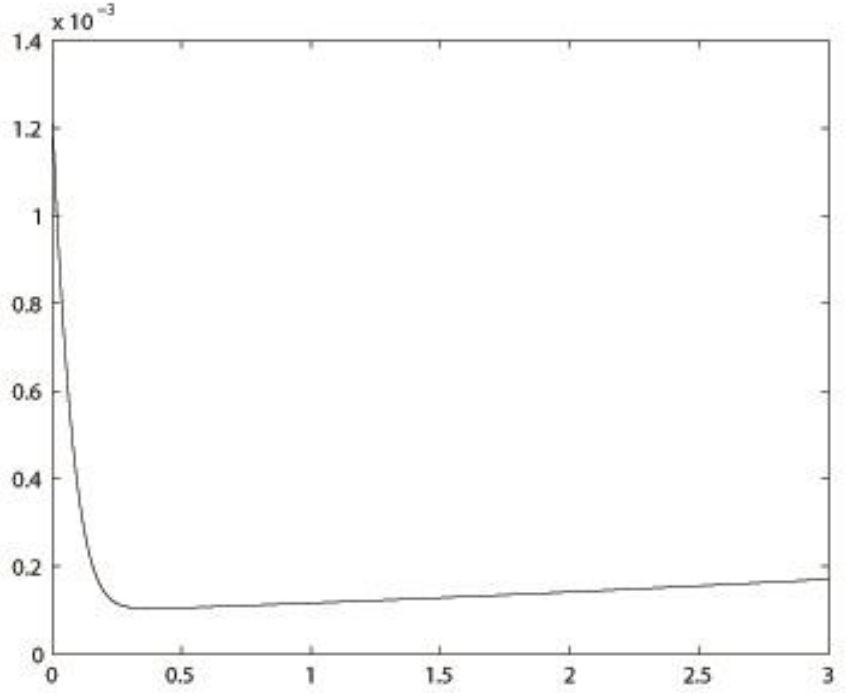

Fig. 1 $x_2(t)$ time-dependence over interval [0, 3]

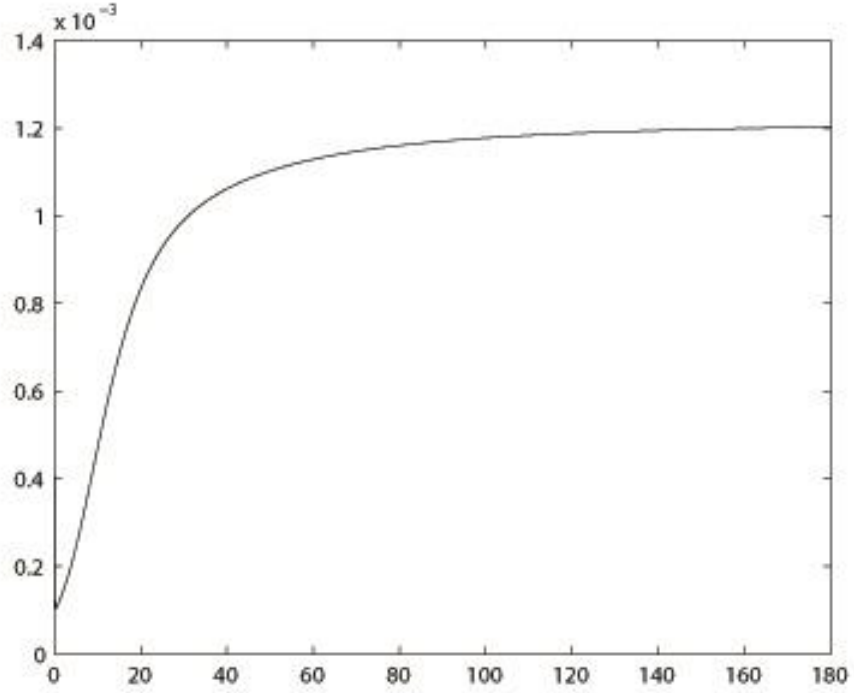

Fig. 2 $x_2(t)$ time-dependence over interval [0, 180]

CHEMICAL AKZO NOBEL PROBLEM

h	***Err***	***Scd***
10^{-2}	$1.6598 \cdot 10^{-5}$	4.7800
10^{-3}	$1.8038 \cdot 10^{-7}$	6.7438
10^{-4}	$1.8231 \cdot 10^{-9}$	8.7392

Table 1 Numerical results for the Chemical Akzo Nobel Problem

- ***h*** is the integration stepsize,
- ***Err*** is the average absolute error,
- ***Scd*** is the average number of significant digits, which was computed by formula

$$\boldsymbol{Scd} = -\log_{10} |\,\text{relative error}\,|. \qquad \textbf{(15)}$$

PENDULUM PROBLEM

The **second test problem** is given by the following system of 5 **nonlinear DAE of index 2 [11]**:

$$\begin{cases} x_1' = x_3, \\ x_2' = x_4, \\ m x_3' = -x_1 y_1, \\ m x_4' = -x_2 y_1 - mg, \\ 0 = x_3 x_1 + x_4 x_2, \end{cases} \qquad \textbf{(16)}$$

where

$$x_1(0) = l, \; x_i(0) = 0, \; 2 \le i \le 4, \; y_1(0) = 0, \; 0 \le t \le \pi.$$

Parameters of the Pendulum problem are

$$m = 98 \cdot 0.4536, \; l = 3.92515344, \; g = 9.80665.$$

[11] Hairer E., Wanner G., Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems. Berlin: Springer-Verlag, 1996.

PENDULUM PROBLEM

h	***Err***	***Scd***
$\pi \cdot 10^{-2}$	$4.4626 \cdot 10^{-1}$	0.3504
$\pi \cdot 10^{-3}$	$4.8694 \cdot 10^{-3}$	2.3125
$\pi \cdot 10^{-4}$	$4.7526 \cdot 10^{-5}$	4.3231

Table 2 Numerical results for the Pendulum Problem

- ***h*** is the integration stepsize,
- **Err** is the average absolute error,
- ***Scd*** is the average number of significant digits.

NUMERICAL RESULTS

From the numerical results given in the tables it follows that **the proposed method has the second accuracy order**. As for the first problem, on increasing the integration stepsize by an order the relative accuracy of the approximate solution increases by two orders.

It is impossible to increase the integration stepsize of the method by an order one more time, since, as it can be seen from Fig. 1, width of the solution transition region is substantially less than 0.5.

For the second problem **it is impossible to make the integration stepsize greater saving the qualitative solution behaviour**. The high relative error on $h = \pi \cdot 10^{-2}$ is possible due to the great integration stepsize chosen.

CONCLUSION

A numerical **non-iterative one-step method is derived**. It is aimed at solving the wide class of the control theory problems, which behaviour is given by system **(1), (2)**. The new method is ***L*-stable** and has the **second accuracy order**.

It is aimed at solving initial value problems for the **DAE systems of index** not exceeding **two**.

Numerical results with the constant integration stepsize show that the formulation of the variable stepsize algorithm will allow to create an efficient solver.

In the prospect it is necessary to add the option of **"freezing"** matrices of derivatives and derive methods of **higher accuracy orders**.

This will increase the efficiency of the methods **and expand the possibilities of (m, k)-methods** for simulating control theory problems (which are often given by initial value and boundary value problems) with high precision.

THANK YOU FOR YOUR ATTENTION