Approximate solving of an inverse problem for a parabolic equation with nonlocal data

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Introduction

We consider the problem of controlling the temperature distribution in the rod, the lateral surface of which is thermally insulated, zero temperature is maintained at the left end, and the change in the amount of heat in the rod is zero. The control is carried out by the unknown initial temperature distribution. The problem is ill-posed in the considered functional spaces, the operator of the problem is non-self-adjoint. An approximate solution is constructed using a regularization method, which consists in replacing the unstable initial problem by a stable problem for the heat conductivity equation with a small parameter in the overdetermination conditions. An exact in order error estimate is obtained for the constructed approximate solution.

Introduction

Let $u(x,t)$ be a solution of the mixed problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \; (0 < x < 1), \tag{1}$$

$$u(0,t) = 0; \tag{2}$$

$$u(x,t_0) = \psi(x) \quad (0 < t_0 < T),$$

$$\int\limits_0^1 u(x,t)dx = 0.$$

Introduction

We consider the following control problem originally formulated by A.A. Samarsky. Given $T > 0$ and $\chi(x) \in W_2^1[0,1]$ such that $\chi(0) = 0$ and $\int\limits_0^1 \chi(x)dx = 0$, we have to minimize the functional

$$J(\varphi) = \int\limits_0^1 |u(x,T;\varphi) - \chi(x)|^2 dx.$$

Integral boundary conditions are encountered, for example, in simulating the process of diffusion of particles in a turbulent plasma, as well as in simulating the process of heat propagation in a thin heated rod, if the law of change in the total amount of heat in the rod is specified.

Introduction

Heat conductivity issues are of particular importance in such areas as aviation and space technology, energetic, metallurgy. Indeed, experimental studies are important as well as full-scale testing of thermal regimes, the creation of effective methods for diagnostics and identification of heat transfer processes judging by the results of experiments and tests.
These methods can be based on the solution of inverse problems of heat transfer, and in some cases, solving inverse problems is really the only possibility to obtain the desired results. Methods of inverse problems are highly informative and allow the researchers to carry out an experiment in conditions close to reality [2].
Using the expansion of a given function into a biorthogonal series in terms of the eigeтfunctions and the associated functions of a non-self-adjoint boundary value problem ([4]), a representation for the solution of the regularized problem is obtained.
The representation for the approximate solution gives the possibility to obtain the inequalities which allow us to estimate the accuracy of the regularization method for solving the inverse problem under study.

Formulation of the retrospective inverse problem

The exact solution of the optimization problem solves the retrospective inverse problem as well

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ (0 < x < 1), \tag{3}$$

$$u(0,t) = 0; \tag{4}$$

$$u(x,T) = \chi(x),$$

$$\int_0^1 u(x,t)dx = 0,$$

$u(x,t_0) = \psi(x) \in L_2[0,1]$ is to be determined.

Formulation of the retrospective inverse problem

Differentiating the integral relation with respect to t we get

$$\int_0^1 u_t(x,t)dx = \int_0^1 u_{xx}(x,t)dx =$$

$$u_x(1,t) - u_x(0,t) = 0.$$

Thus, the inverse retrospective problem (3) -(4) is equivalent to the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ (0 < x < 1), \tag{5}$$

$$u(0,t) = 0; \tag{6}$$

$$u(x,T) = \chi(x),$$

$$u_x(1,t) - u_x(0,t) = 0,$$

$$u(x,t) \in C([t_0;T]; W_2^{2,0}[0;l]) \cap C^1((t_0;T); L_2[0;l]),$$

$u(x,t_0) = \psi(x)$ $(0 < t_0 < T)$ is to be determined.

Formulation of the retrospective inverse problem

The problem (3)- (4) is improperly posed.
Suppose that for the given $\chi(x) \in L_2[0,1]$ the problem ((2)- (3) has the exact solution, $u(x,t)$ satisfies (3)- (4) . Suppose that $u(x,t)$ solves the equation (3)- (4) for $t \in (0,T)$.
Denote $\varphi(x) = u(x,0)$. We suppose that the exact solution of the inverse problem belongs to the set

$$M = \{\psi(x) : \|\varphi\|_{L_2[0,1]}\|^2 \leq r^2\},$$

but instead of the exact values if $\chi(x)$ we know $\delta-$ approximations χ_δ and the error level $\delta > 0$ such that $\|\chi_\delta - \chi\| < \delta$. We have to construct a stable approximate solution of the problem and to estimate the error of the approximate solution from the input data of the problem

The exact solution of the inverse problem

Consider the non-self-adjoint boundary value problem

$$X'(x) + \lambda X(x) = 0 \ (0 < x < 1), \tag{7}$$

$$X(0,t) = 0; \quad X'(0) = X'(1). \tag{8}$$

The problem (7) -(8) has the eigenvalues $\lambda_k = (2\pi k)^2, \quad k = 0, 1, ...$
Consider the system of the eigenfunctions and the associated functions for the problem (7) -(8)

$$X_0(x) = x, \tag{9}$$

$$X_{2k-1} = x\cos(2\pi kx),$$

$$X_{2k} = \sin(2\pi kx) \ \ (k = 1, 2, ...).$$

The normed eigenfunctions and the associated functions for the adjoint problem are

$$Y_0(x) = 2, \tag{10}$$

$$Y_{2k-1} = 4\cos(2\pi kx),$$

$$Y_{2k} = 4(1-x)\sin(2\pi kx)$$

The exact solution of the inverse problem

The sets of functions (9) and (10) form a biorthogonal system on the segment $[0, 1]$, so for any numbers i and j

$$(X_i, Y_j) = \int\limits_0^1 X_i(x)Y_j(x)dx = \delta_{ij},$$

where δ_{ij} stands for Kronecker Delta. So, an arbitrary function $\varphi(x) \in L_2[0, 1]$ can be expanded into the series

$$\varphi(x) = \varphi_0 X_0(x) + \sum_{k=1}^{\infty}(\varphi_{2k}X_{2k}(x) + \varphi_{2k-1}X_{2k-1}(x)),$$

the coefficients of the series can be calculated as follows

$$\varphi_0 = \int\limits_0^1 \varphi(x)Y_0(x)dx;$$

$$\varphi_{2k} = \int \varphi(x)Y_{2k}(x)dx; ;$$

The exact solution of the inverse problem

The following theorem holds [4]

Theorem

Let $\varphi(x)$ be a continuously differentiable function which meets the conditions

$$\varphi(0) = 0, \quad \varphi'(0) = \varphi'(1).$$

Then the mixed problem (1) -(2) *has the unique solution of the form*

$$u(x,t) = \sum_{k=1}^{\infty} \left[\varphi_{2k} X_{2k}(x)\right] e^{-\lambda_k t} + \tag{11}$$

$$\sum_{k=1}^{\infty} \left[\varphi_{2k-1}(X_{2k-1}(x) - 2\sqrt{\lambda_n} t X_{2k}(x)\right] e^{-\lambda_k t}$$

The exact solution of the inverse problem

Let the given function $\chi(x) \in L_2[0,1]$ have the expansion

$$\chi(x) = \sum_{k=1}^{\infty} [\chi_{2k} X_{2k}(x)] + \tag{12}$$

$$\sum_{k=1}^{\infty} \left[\chi_{2k-1}(X_{2k-1}(x) - 2\sqrt{\lambda_n} t X_{2k}(x)) \right].$$

Therefore, for any function $\chi(x) \in L_2[0,1]$ for which the inverse problem has a solution, this solution can be represented as

$$\psi(x) = \sum_{k=1}^{\infty} [\chi_{2k} X_{2k}(x) + \chi_{2k-1}(X_{2k-1}(x)] . e^{\lambda_k (T-t_0)} + \tag{13}$$

$$\sum_{k=1}^{\infty} \left[\chi_{2k} X_{2k}(x) - 2\sqrt{\lambda_n} T X_{2k}(x)) \right] . e^{\lambda_k (T-t_0)}.$$

The regularization method

Instead of the unstable problem (2))- (3) we consider a problem with a small parameter in the overdetermination conditions that is we should recover the function $\varphi_\varepsilon(x) = u_\varepsilon(lx, 0)$, where $\varphi_\varepsilon(t) \in L_2[0,1)$, $u_\varepsilon(x,t)$ satisfies the conditions

$$\frac{\partial u_\delta^\varepsilon}{\partial t} = \frac{\partial^2 u_\delta^\varepsilon}{\partial x^2}, \ (0 < x < l), \tag{14}$$

$$u_\delta^\varepsilon(0,t) = 0; \quad \varepsilon u_\delta^\epsilon(x,0) + u_\delta^\epsilon(x,T) = \chi_\delta(x); \tag{15}$$

$$\int_0^1 u_\delta^\varepsilon(x,t)dx = 0.$$

Here $\varepsilon > 0$ is the regularization parameter, we should. descry the appropriate relation $\varepsilon = \varepsilon(\delta)$.

Let $u_\delta^\varepsilon(x,t)$ be the solution to the problem (14) -(15) . We consider the function

$$\psi_\delta^\varepsilon(x) = u_\delta^\varepsilon(x, t_0)$$

as an approximate solution to the problem (1).

The regularization method

Differentiating the integral relation in (15) with respect to t we get

$$\int_0^1 \left(\varepsilon \frac{\partial^2 u_\delta^\varepsilon}{\partial t^2} + \frac{\partial u_\delta^\varepsilon}{\partial t}\right) dx = \int_0^1 \frac{\partial^2 u_\delta^\varepsilon}{\partial x^2}(x,t)dx =$$

$$\frac{\partial u_\delta^\varepsilon}{\partial x}(1,t) - \frac{\partial^2 u_\delta^\varepsilon}{\partial x^2}(0,t) = 0.$$

Thus, the regularized mixed boundary problem (14) -(15) is equivalent to the problem

$$\frac{\partial u_\delta^\varepsilon}{\partial t} = \frac{\partial^2 u_\delta^\varepsilon}{\partial x^2} \ (0 < x < l), \tag{16}$$

$$u_\delta^\varepsilon(0,t) = 0; \tag{17}$$

$$\varepsilon u_\delta^\epsilon(x,0) + u_\delta^\epsilon(x,T) = \chi_\delta(x),$$

$$\frac{\partial u_\delta^\varepsilon}{\partial x}(1,t) - \frac{\partial^2 u_\delta^\varepsilon}{\partial x^2}(0,t) = 0.$$

The solution to the regularized problem

Solving a regularized problem by the Fourier method, we make sure that the regularized problem (16) -(17) has the unique solution of the form

$$u_\delta^\epsilon(x,t) = \sum_{k=1}^{\infty} \left[\varphi_{2k}^{\varepsilon,\delta} X_{2k}(x) \right] e^{-\lambda_k^2 t} + \tag{18}$$

$$\sum_{k=1}^{\infty} \left[\varphi_{2k-1}^{\varepsilon,\delta} (X_{2k-1}(x) - 2\sqrt{\lambda_n} t X_{2k}(x)) \right] e^{-\lambda_k^2 t}.$$

Substituting the relation (18) into the overdetermination condition with a small parameter, we find the relations between the coefficients of the expansion for the solution and for the given function

$$\psi_{2k}^{\varepsilon,\delta} = \frac{\psi_{2k-1}^{\varepsilon,\delta} e^{\lambda_k T}}{1+\varepsilon e^{\lambda_k T}} + \frac{2\sqrt{\lambda_n} t 2\sqrt{\lambda_n} T \chi_{2k-1}^{\varepsilon,\delta} e^{\lambda_k T}}{(1+\varepsilon e^{\lambda_k T})^2}, \tag{19}$$

$$\psi_{2k-1}^{\varepsilon,\delta} = \frac{\chi_{2k-1}^{\varepsilon,\delta} e^{\lambda_k T}}{1+\varepsilon e^{\lambda_k T}}.$$

The solution to the regularized problem

Therefore, for any function $\chi(x) \in L_2[0,1]$ the solution of the regularized problem can be represented as

$$\psi_\delta^\epsilon(x) = \sum_{k=1}^{\infty} \left[\frac{\chi_{2k}^{\varepsilon,\delta} X_{2k}(x) e^{\lambda_k (T-t_0)}}{1+\varepsilon e^{\lambda_k T}} \right] + \tag{20}$$

$$+ \sum_{k=1}^{\infty} \left[\frac{2\sqrt{\lambda_n} T \chi_{2k-1}^{\varepsilon,\delta} X_{2k}(x) e^{\lambda_k (T-t_0)}}{(1+\varepsilon e^{\lambda_k T})^2} \right] +$$

$$\sum_{k=1}^{\infty} \left[\frac{\chi_{2k-1}^{\varepsilon,\delta} X_{2k-1}(x) e^{\lambda_k (T-t_0)}}{1+\varepsilon e^{\lambda_k T}.} \right] -$$

$$\sum_{k=1}^{\infty} \left[\frac{2\sqrt{\lambda_n} T \chi_{2k-1}^{\varepsilon,\delta} X_{2k-1}(x) e^{\lambda_k (T-t_0))}}{1+\varepsilon e^{\lambda_k T}} \right].$$

The error estimate for the approximate solution

Let $\varphi(x)$ be an exact solution to the problem (1). We use the value

$$\Delta(\varepsilon,\delta)=\sup\{\|\psi_\delta^\varepsilon-\psi\|:\psi\in M;\|\chi-\chi_\delta\|\le\delta\}$$

to estimate the accuracy of the approximate solution to the problem (1). constructed by means of the regularization method (14) -(15) on the set M. Here we should choose the relation between ε and δ to minimize the error of the obtained approximate solution (quasi-optimal choice of the regularization parameter). We use the obvious estimate

$$\Delta(\varepsilon,\delta)\le\Delta_1(\varepsilon,)(\varepsilon,\delta)+\Delta_2(\varepsilon,\delta),$$

where

$$\Delta_2(\varepsilon,\delta)=\sup_{\|\chi-\chi_\delta\|\le\delta}\|\psi_\delta^\varepsilon-\psi^\epsilon\|;\quad \Delta_1(\varepsilon)=\sup_{\varphi\in M}\|\psi_\varepsilon-\psi\|.$$

Here ψ_ε is the solution of the auxiliary problem (14) -(15) for the precisely defined initial condition.

The error estimate for the approximate solution

We estimate the value of $\Delta_2(\varepsilon, \delta)$. The solutions to the auxiliary problem (14) -(15) for the exactly given and the approximately given initial data satisfy the condition

$$\psi_\delta^\epsilon(x) - \psi^\epsilon(x) = \sum_{k=1}^{\infty} \left[\frac{(\chi_{2k}^{\varepsilon,\delta} - \chi_{2k}^{\varepsilon} e^{\lambda_k (T-t_0)}}{1 - 2\lambda_k T + \varepsilon e^{\lambda_k T}} X_{2k}(x) \right] e^{-\lambda_k^2 T} + \qquad (21)$$

$$\sum_{k=1}^{\infty} \left[\frac{\chi_{2k-1}^{\varepsilon,\delta} - \chi_{2k-1}^{\varepsilon}) e^{\lambda_k (T-t_0)}}{1 + \varepsilon e^{\lambda_k T}} (X_{2k-1}(x) - 2\sqrt{\lambda_n} t X_{2k}(x)) \right].$$

The error estimate for the approximate solution

We estimate the fraction

$$F(\lambda) = \frac{e^{\lambda(T-t_0)}}{1+\varepsilon e^{\lambda T}}.$$

Denote $s = e^{\lambda T}$, $s \geq 1$. Calculate the maximal value of function

$$F(s) = \frac{s^{\frac{T-t_0}{T}}}{1+\varepsilon s}.$$

The critical point of $F(s)$ is $s_0 = \frac{T-t_0}{t_0}\frac{1}{\varepsilon}$. Further,

$$F(1) = \frac{1}{1+\varepsilon}, \quad F(s_0) = \frac{C_1}{\varepsilon}, \; C_1 = \left(\frac{T-t_0}{t_0}\right)^{\frac{T-t_0}{T}} \frac{t_0}{T}, \quad \lim_{s\to+\infty} F(s) = 0.$$

Hence,

$$\alpha = \sup_{\lambda \geq 0} F(\lambda) = \frac{C_1}{(\varepsilon)^{\frac{t_0}{T}}},$$

The error estimate for the approximate solution

Estimate the fraction

$$\Phi(\lambda) = \frac{e^{\lambda(T-t_0)}}{1 - 2\sqrt{\lambda}T + \varepsilon e^{\lambda T}}. \tag{22}$$

We estimate the denominator of the fraction from above. Denote $s = \sqrt{\lambda}$ and write the denominator in the form

$$f(s) = 1 - 2sT + \varepsilon e^{s^2 T}.$$

As $1 - 2sT < 0$, $\varepsilon e^{s^2 T} > 0$ for $s > \frac{1}{2T}$, then $f(s) > 0$ for $s > \frac{1}{2T}$. Consider the derivative

$$f'(s) = -2T + 2\varepsilon T s e^{s^2 T}. \tag{23}$$

As $f'(0) = -2T$, $\lim\limits_{s \to +\infty} = +\infty$, then there exists a critical point $s_0 = s_0(\varepsilon)$ of the function $f(s)$. The equality (23) imply that $d_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Hence, for every $S > 0$ we can find ε_0 such that for all $\varepsilon < \varepsilon_0$

$$s_0(\varepsilon) < S.$$

The error estimate for the approximate solution

Take into account that

$$\frac{\sqrt{\lambda}T}{e^{\lambda T}} \to 0$$

as $\lambda \to \infty$ and for $\varepsilon < \varepsilon_0$ choose $S > 0$ such that for all $\lambda > S$

$$2\frac{\sqrt{\lambda}T}{e^{\lambda T}} < \varepsilon.$$

Consequently, the value of the function in the critical point meets the inequality

$$f'(s_0) = -1 - 2Ts_0 + 2\varepsilon e^{s_0^2 T} \leq 1 + \frac{\varepsilon}{2} e^{s_0^2 T}. \tag{24}$$

It follows from the inequality (24) that for all $\lambda > 0$ and for sufficiently small $\varepsilon > 0$ the following inequality holds

$$\beta = \sup_{\lambda \geq 0} \Phi(\lambda) \leq \frac{e^{\lambda(T-t_0)}}{1 + \varepsilon e^{\lambda T}} \leq \frac{C_1}{(\varepsilon)^{\frac{t_0}{T}}},$$

$$C_1 = \left(\frac{T - t_0}{t_0}\right)^{\frac{T-t_0}{T}} \frac{t_0}{T}. \tag{25}$$

The error estimate for the approximate solution

Estimate the fraction

$$G(\lambda) = \frac{\sqrt{\lambda} e^{\lambda(T-t_0)}}{1+\varepsilon e^{\lambda T}}, \quad \lambda \geq 0.$$

Take into consideration that for any $\tau > 0$, $\lambda > 0$

$$\sqrt{\lambda} \leq e^{\lambda \tau}.$$

Consequently,

$$G(\lambda) \leq \frac{e^{\lambda(T+\tau-t_0)}}{1+\varepsilon e^{\lambda T}}. \tag{26}$$

Estimating the right-hand side of the inequality (26) in a similar way as we have obtained the the estimation for the function $F(\lambda)$, we prove that

$$G(\lambda) \leq \frac{C_2}{(\varepsilon)^{\frac{t_0-\tau}{T}}}, \tag{27}$$

As $\tau > 0$ is an arbitrary number, then the inequality (27) implies

$$\gamma = \sup_{\lambda \geq 0} G(\lambda) \leq \frac{C_2}{(\varepsilon)^{\frac{t_0}{T}}}, \tag{28}$$

The error estimate for the approximate solution

Estimate the value Δ_1. Consider the equality

$$\psi^{\epsilon}(x) - \psi(x) = \sum_{k=1}^{\infty}\left[\frac{\varepsilon\chi 2k X_{2k}(x)e^{\lambda_k(T-t_0)}}{1+\varepsilon e^{\lambda_k T}}\right] + \tag{30}$$

$$+\sum_{k=1}^{\infty}\left[\frac{\varepsilon 2\sqrt{\lambda_n}T\chi_{2k-1}X_{2k}(x)e^{\lambda_k(T-t_0)}}{(1+\varepsilon e^{\lambda_k T})^2}\right] +$$

$$\sum_{k=1}^{\infty}\left[\frac{\varepsilon\chi_{2k-1}X_{2k-1}(x)e^{\lambda_k(T-t_0)}}{1+\varepsilon e^{\lambda_k T}}\right] -$$

$$\sum_{k=1}^{\infty}\left[\frac{\varepsilon 2\sqrt{\lambda_n}T\chi_{2k-1}X_{2k-1}(x)e^{\lambda_k(T-t_0))}}{1+\varepsilon e^{\lambda_k T}}\right].$$

The error estimate for the approximate solution

The definition of the set M imply the inequality

$$\sum_{n=1}^{\infty} \varphi_{2k}^2 + \sum_{n=1}^{\infty} \varphi_{2k-1}^2 \leq r^2.$$

Consequently, for every k, $k = 1, 2, \dots$

$$|\chi_{2k} e^{\lambda_k T}| \leq r, \tag{31}$$

$$2\sqrt{\lambda} T |\chi_{2k-1} e^{\lambda_k T}| \leq 2r.$$

The inequalities (30) and (31) imply

$$\|\psi_\varepsilon(x) - \psi(x)\|_{L_2[0,1]} \leq C_2 \sum_{k=1}^{\infty} (\psi_k^\varepsilon - \psi_k)^2 \tag{32}$$

$$\leq 4 C_2 \varepsilon (\alpha + \gamma).$$

Here

$$\alpha = \sup_{\lambda \geq 0} \frac{e^{\lambda(T-t_0)}}{1 + \varepsilon e^{\lambda T}}, \quad \gamma = \sup_{\lambda \geq 0} \frac{\sqrt{\lambda} e^{\lambda(T-t_0)}}{1 + \varepsilon e^{\lambda T}}$$

The error estimate for the approximate solution

Taking into consideration that

$$\alpha \le \gamma \le \frac{C}{(\varepsilon)^{\frac{t_0}{T}}}$$

we obtain from (32)

$$\|\psi_\varepsilon(x) - \psi(x)\|_{L_2[0,1]} \le \frac{C_3}{(\varepsilon)^{\frac{T-t_0}{T}}} \tag{33}$$

Therefore, we have the estimates

$$\Delta_1(\varepsilon) \le C\varepsilon^{\frac{T-t_0}{T}},$$

$$\Delta_2(\varepsilon, \delta) \le \frac{C\delta}{(\varepsilon)^{\frac{t_0}{T}}}.$$

Consequently,

$$\Delta(\varepsilon, \delta) \le C(\varepsilon)^{\frac{T-t_0}{T}\cdot} + \frac{C\delta}{(\varepsilon)^{\frac{t_0}{T}}}. \tag{34}$$

The error estimate for the approximate solution

Choose the relation $\varepsilon = \varepsilon^*(\delta)$ using M,M.Lavrent'ev schene, that is we choose the regularization parameter from the equality

$$\Delta_1(\varepsilon) = \Delta_2(\varepsilon, \delta)$$

It is known that such choice of the regularization parameter ensures the optimal order for the estimate (34) [1]. We obtain the relation

$$\varepsilon^{\frac{T-t_0}{T}\cdot} = \frac{C\delta}{\varepsilon^{\frac{t_0}{T}}}. \tag{35}$$

The equality (35) implies

$$\varepsilon^*(\delta) = C_3\delta. \tag{36}$$

Here C_3 is a constant which does not depend on δ.
Given the resulting equality (36) we obtain the uniform error estimate for the regularized solution

$$\Delta(\varepsilon^*, \delta) \le C_4(\delta)^{\frac{T-t_0}{T}}.$$

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Thank you for your attention!