Processing Measurement Uncertainty: From Intervals and p-Boxes to Probabilistic Nested Intervals

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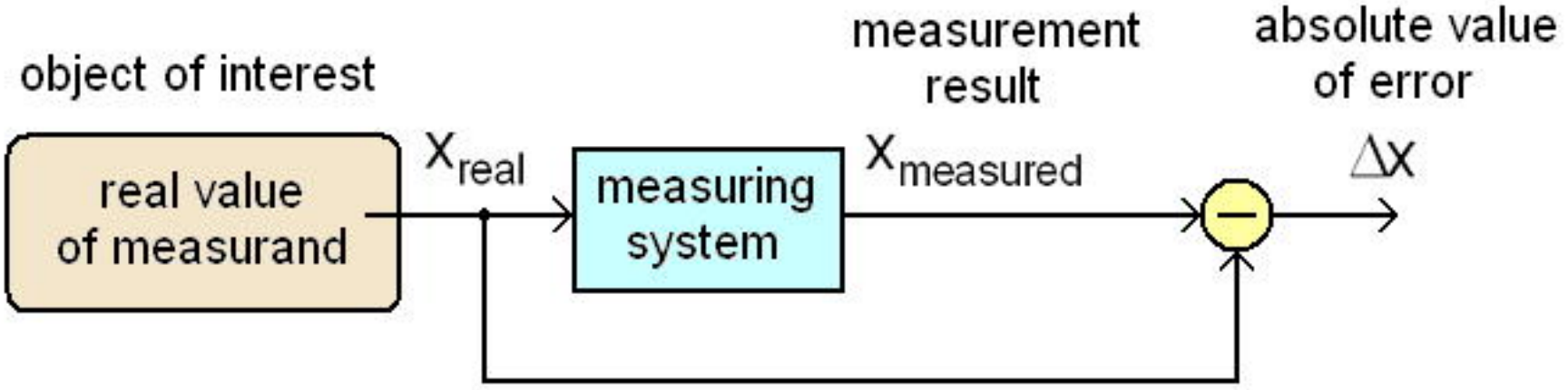
15th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetics and Verified Numerics

September, 23-28, 2012.

Introduction: measurements

→ **The main property of any measurement result is its uncertainty or error**. It is the main quality parameter for performed measurement.

→ Let us measure some voltage quantity x_{real}. Let us receive $x_{measured}$ = 1.05 V from measuring system. **Is it close to the real value** x_{real}? To answer how accurate it is, one has to estimate its absolute error $\Delta x = x_{measured} - x_{real}$.

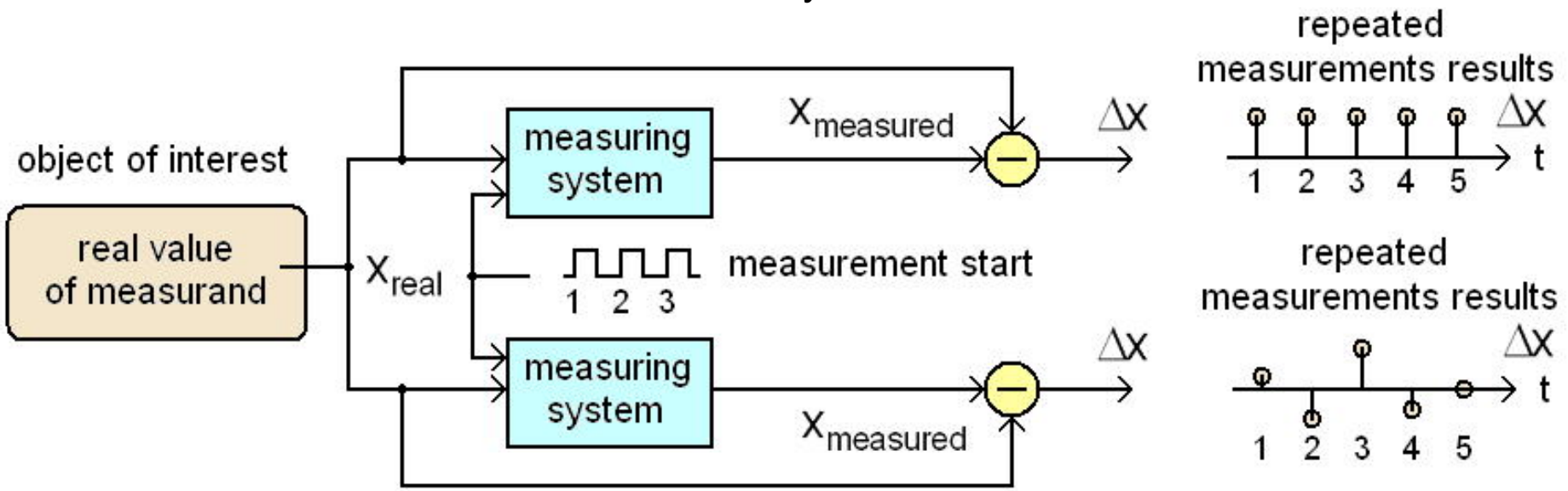


Introduction: measurements

→ **We never know the measurand real value** x_{real}.

We never know the error value Δx. The only thing we can do is to use interval $[\underline{\Delta x}, \overline{\Delta x}]$ of its possible values. Its bounds can be retrieved from technical documentation for used measuring instrument.

→ The error of measurement result can have different nature: it can be **systematic** $\Delta_{syst}x$ or **random** $\Delta_{rand}x$ or mixed.



Measurements results processing

→ **What do we have from technical documentation?** In almost all practice situations we only have two intervals:

$$\Delta_{syst}x \in \left[\underline{\Delta_{syst}x},\ \overline{\Delta_{syst}x}\right] \text{ for systematic component,}$$

$$\text{Prob}\left(\Delta_{rand}x \in \left[\underline{\Delta_{rand}x},\ \overline{\Delta_{rand}x}\right]\right) = 0.95 \text{ for random one.}$$

Usually $\overline{\Delta_{rand}x} = -\underline{\Delta_{rand}x} = k \cdot \sigma_x$, where σ_x is a standard deviation of error random component.

→ **Errors of measurement results are usually small. How accurate should borders of these intervals be?** In metrology we always have to round final calculations results.

incorrect		correct
$x = 1.06\ V,$ $\overline{\Delta x} = -\underline{\Delta x} = 0.09\ V$	⇒	$x = 1.1\ V,$ $\overline{\Delta x} = -\underline{\Delta x} = 0.1\ V$

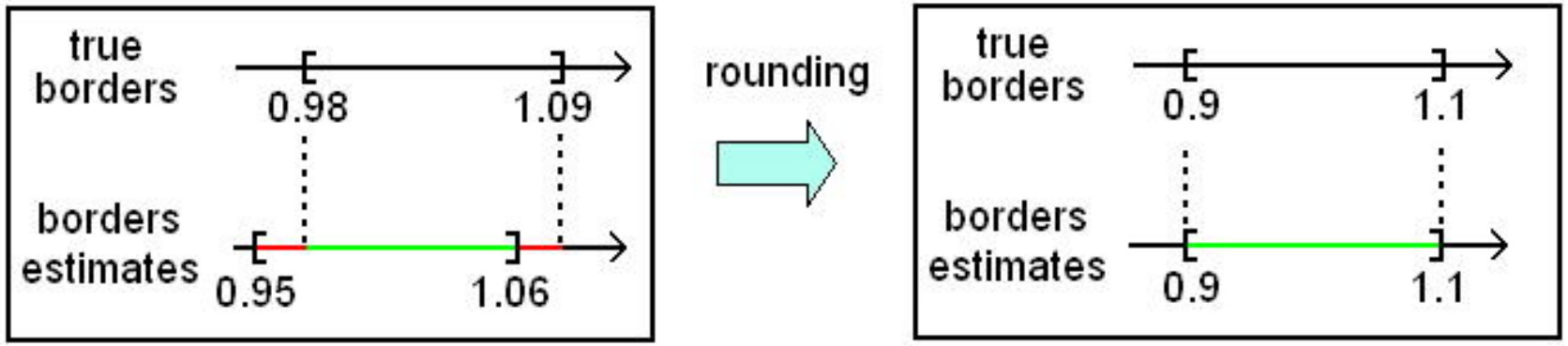
Measurements results processing

→ **Metrological case is specific.**

Classical approaches for uncertainty propagation always provide bounds $J[\Delta x]$ for estimated interval $\left[\underline{\Delta x}, \overline{\Delta x}\right]$ that guarantees its coverage: $J \supseteq \left[\underline{\Delta x}, \overline{\Delta x}\right]$.

As a conclusion J is almost always overestimated, sometimes catastrophically.

→ **In metrology we can allow J to be slightly over- or even slightly underestimated because of results' rounding.**



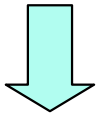
Measurements results processing

→ **Conclusion. Linearization can be used.**

Let $y = f(x_1, ..., x_n)$ be a function to process the measurement results $x_1, ..., x_n$. Then

$$\Delta y \approx \sum_{i=1}^{n} \frac{\partial f(x_1, ..., x_n)}{\partial x_i} \cdot \Delta x_i$$

Function f is determined by computer program. To obtain its partial derivatives we can use **automatic differentiation** technique.



We can take into consideration only linear operations with measurement errors for its arithmetic construction.

What mathematical framework to use?

→ **We are not allowed to process random and systematic error components in one way in metrology**:

$$y = f(x_1, ..., x_n)$$

for independent

$$x_1, ..., x_n$$

$$\Delta_{syst} y \approx \sum_{i=1}^{n} \frac{\partial f(x_1, ..., x_n)}{\partial x_i} \cdot \Delta_{syst} x_i$$

$$\sigma_y \approx \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f(x_1, ..., x_n)}{\partial x_i} \cdot \sigma_{x_i} \right)^2}$$

→ **What mathematical framework should we use to process measurement errors?**

Let us average some repeated measurements results $x_1, ..., x_n$ for the same quantity. If all Δx_i are from interval $\left[\underline{\Delta x}, \overline{\Delta x}\right]$ then using interval arithmetic provides us the following results

$$\Delta y = \frac{1}{n} \cdot \sum_{i=1}^{n} \Delta x_i$$

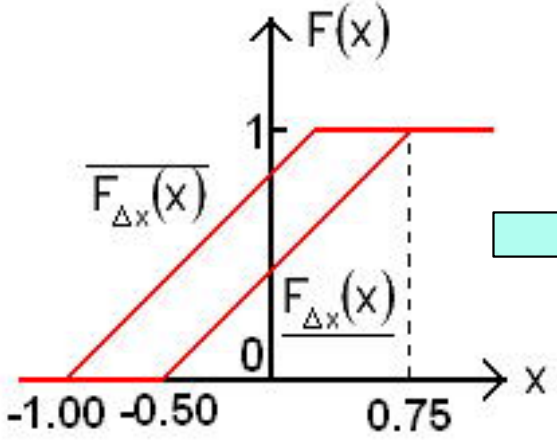
$$\Delta y \in \left[\underline{\Delta x}, \overline{\Delta x}\right]$$

we'll never get $\Delta y \in \left[\frac{\underline{\Delta x}}{\sqrt{n}}, \frac{\overline{\Delta x}}{\sqrt{n}}\right]$!

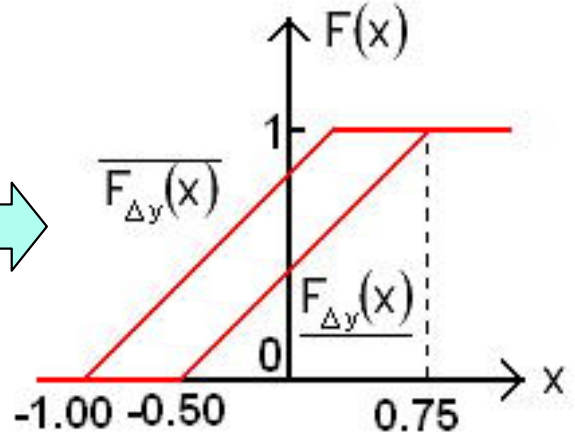
What mathematical framework to use?

→ **Conclusion. Classical interval techniques (Moore's arithmetic, affine arithmetic etc) can be used for systematic errors propagating (but, of course, if we use interval methods for random errors, we get a drastic overestimation).**

Let us average some repeated measurements results $x_1, ..., x_n$. If all Δx_i distribute with cdf inside p-box $\left[\underline{F_{\Delta x}(x)}, \overline{F_{\Delta x}(x)}\right]$ then using p-boxes techniques with no assumption about dependence provides us the following results



$$\Delta y = \frac{1}{n} \cdot \sum_{i=1}^{n} \Delta x_i$$



the same one!

What mathematical framework to use?

→ **P-boxes framework can not be used for random error processing as an universal tool because there isn't usually enough information to construct p-boxes for single measurements results.**

→ **We can introduce new instance for error propagating through linear calculations.**

Let us use the tuple $\left\langle \overline{\Delta_{syst}x}, \overline{\sigma_x} \right\rangle$. We can determine linear operations easily:

$$\left\langle \overline{\Delta_{syst}x_1}, \overline{\sigma_{x_1}} \right\rangle \pm \left\langle \overline{\Delta_{syst}x_2}, \overline{\sigma_{x_2}} \right\rangle = \left\langle \overline{\Delta_{syst}x_1} + \overline{\Delta_{syst}x_2}, \sqrt{\overline{\sigma_{x_1}^2} + \overline{\sigma_{x_2}^2}} \right\rangle,$$

$$c \cdot \left\langle \overline{\Delta_{syst}x}, \overline{\sigma_x} \right\rangle = \left\langle |c| \cdot \overline{\Delta_{syst}x_1}, |c| \cdot \overline{\sigma_x} \right\rangle.$$

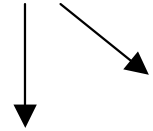
The final interval for error will be of form $\pm\left(\overline{\Delta_{syst}x} + k \cdot \overline{\sigma_x}\right)$.

The question is what value of k we should to choose.

What mathematical framework to use?

→ **In metrology the following result is known [P. V. Novitsky, M. A. Zemelman, V. Ya. Kreinovich]: for the wide family of distributions come from measurement data**

$$\textbf{Prob}\left(\Delta_{\textbf{rand}}\mathbf{x} \in \left[-\mathbf{k}\cdot\boldsymbol{\sigma}_{\mathbf{x}},\ +\mathbf{k}\cdot\boldsymbol{\sigma}_{\mathbf{x}}\right]\right) = \mathbf{0.9}, \quad \textbf{if } \mathbf{k} \in \textbf{[1.55, 1.65]}$$



frequent

subjective

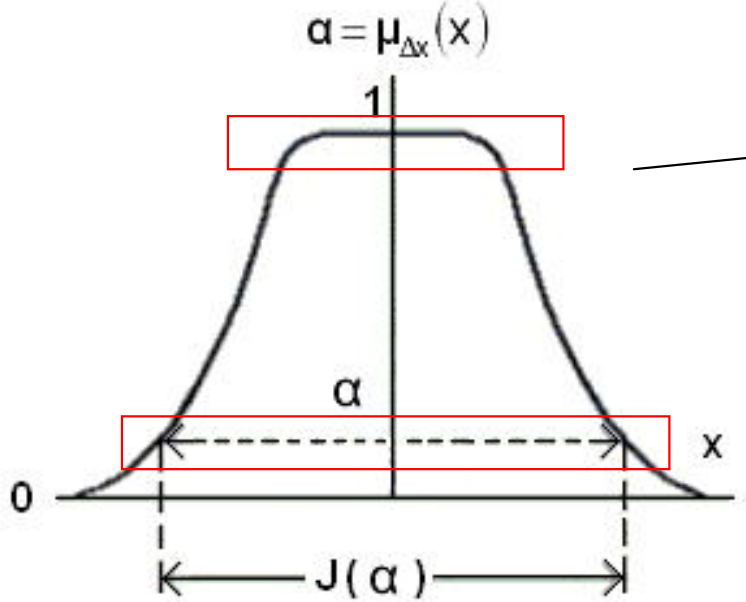
→ **How to take into account the case of expert's estimates?** We can naturally introduce probabilistic nested interval as unified representation for measurement error instead of eclectic tuple.

It is 1-parameter set of intervals $\{J(\alpha)\}$, where $0 \le \alpha \le 1$ is a probability-like measure, such that

$$J(\alpha_1) \supseteq J(\alpha_2) \text{ if } \alpha_2 \ge \alpha_1$$

Fuzzy nested intervals

→ **How does this set represent characteristics of error components?**



J(1) represents interval characteristic for systematic error

J(α) represents interval characteristic for total error (systematic plus random) for probability p = 1-α.

→ **Operations with probabilistic nested intervals are introduced as it is accepted in fuzzy theory.**

$$\mu_{\Delta x_1 \circ \Delta x_2}(z) = \sup_{x_1 \circ x_2 = z} \left\{ T\left(\mu_{\Delta x_1}(x_1), \mu_{\Delta x_2}(x_2)\right) \right\}$$

$$T : [0,1] \to [0,1]$$

$$T(a,b) = T(b,a)$$

$$T(a_1,b_1) \le T(a_2,b_2), \quad \begin{matrix} a_1 \le a_2 \\ b_1 \le b_2 \end{matrix}$$

$$T(T(a,b),c) = T(a,T(b,c))$$

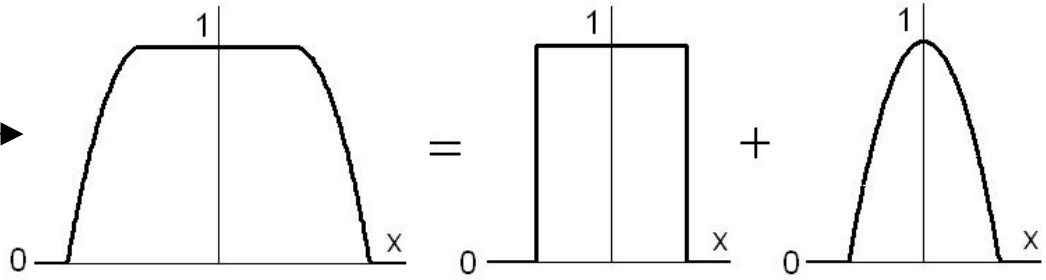
$$T(a,1) = a$$

Fuzzy nested intervals

→ It is easy to show that **there is continuum of different triangular norms that can produce such rules for two scale parameters**. For each of them the form of membership function will be different. What one should we choose?

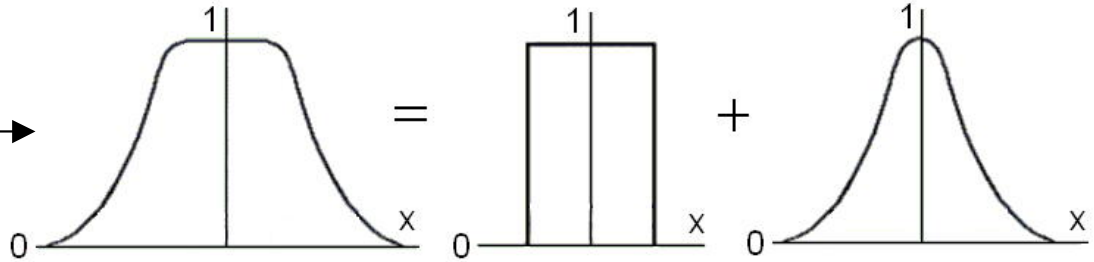
$$\mu_{\Delta x_1 \circ \Delta x_2}(z) =$$

$$\sup_{x_1 \circ x_2 = z} \left\{ \max \left\{ 0, \begin{matrix} \mu_{\Delta x_1}(x_1) + \\ \mu_{\Delta x_2}(x_2) - 1 \end{matrix} \right\} \right\} \rightarrow$$



$$\sup_{x_1 \circ x_2 = z} \left\{ \min \left\{ \mu_{\Delta x_1}(x_1), \mu_{\Delta x_2}(x_2) \right\} \right\} \rightarrow$$ no variant exists

$$\sup_{x_1 \circ x_2 = z} \left\{ \mu_{\Delta x_1}(x_1) \cdot \mu_{\Delta x_2}(x_2) \right\} \rightarrow$$

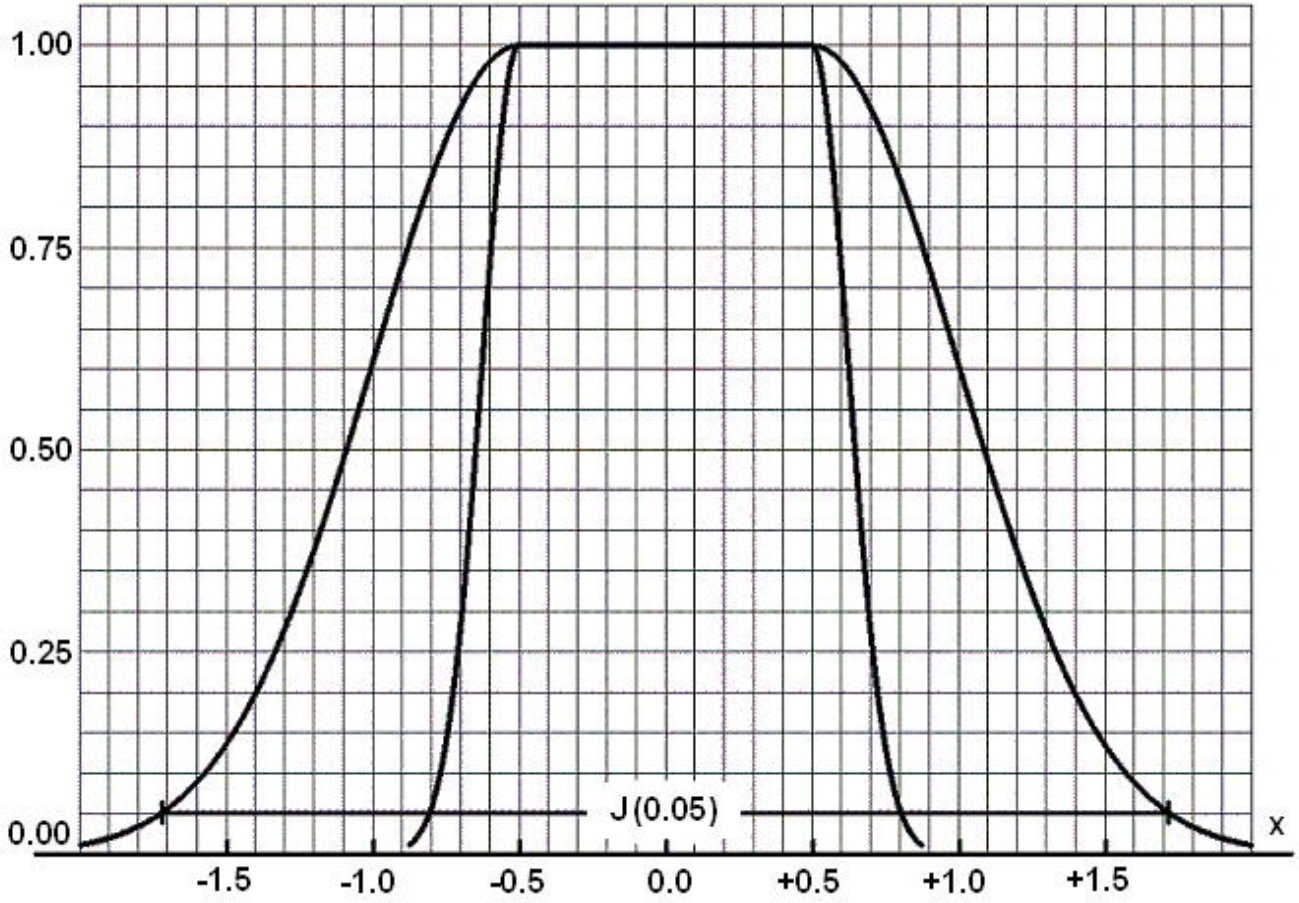


Last variant is the closest to probabilistic character of $\{J(\alpha)\}$.

Fuzzy nested intervals

→ **In this case we can easily process measurement results for the case of linear calculations.**

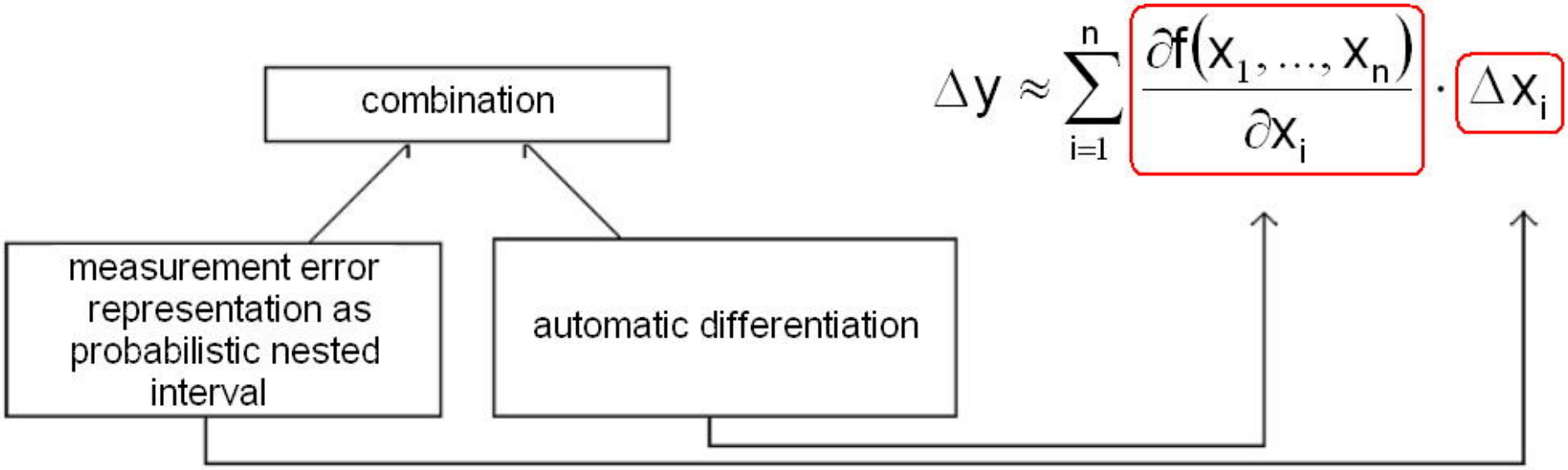
Let us average n = 16 repeated measurements results of one quantity, all Δx_i are represented by the same probabilistic nested interval.



The latter sides for the result of averaging become narrower in $\sqrt{n}$ times as it should be for the random error. J(1) is processed with no changes as it should be for the systematic error.

Conclusion

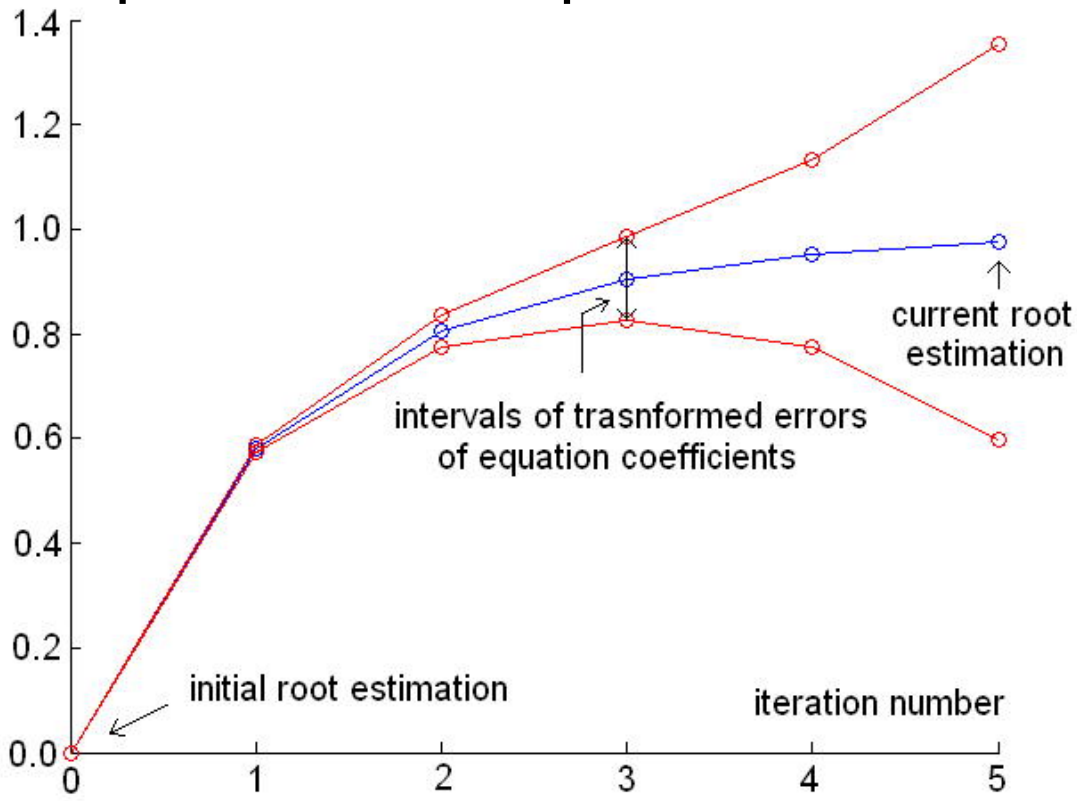
→ **We can process measurement data fast and easily in full correspondence to metrological norms and rules if we will use the combination of automatic differentiation and probabilistic nested interval arithmetic.**



This combination can be easily programmed. Special library was written in C++ for linking with user projects and numerous tests were performed with it.

Application example

Let us solve nonlinear equation by iterative procedure (Newton method). If input data is inaccurate when should we stop the iterative process?



Let the equation be

$$\exp(k \cdot x) = \lambda \cdot x$$

with coefficients

$$\lambda = 2.72 \pm 0.01$$

$$k = 1.00 \pm 0.01$$

The equation may have 2 real roots, only 1 root or no roots at all.

Let x_i be the i-th root estimation. We propose to stop iteration process when the following inequality begins to hold

$$\left| x_{i+1} - x_i \right| < \left\| J_{x_{i+1}}(0.1) \right\| - \left\| J_{x_i}(0.1) \right\|$$

Thank you for attention!