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On Affinity of Physical Processes of Computing and Measurements

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Main points

Abstract

The subject of this development is an algorithm, originally built for natural measurements.

This development is devoted to substantiate extending of the field of application of this algorithm to the area of numerical (instrumental) computation.

Our development based on the postulate that is dynamic generalization of the uncertainty relation (Dynamic UR) of the classical physics.

This postulate had explained the natural experiment that could not be explained before.

The postulate has a corollary: the physical processes of natural measurements and numerical (instrumental) calculations have a single structure.

Numerical (instrumental) experiments confirm the unity of the physical structure of measurement and calculation.

Topicality

The advent of computers became a cause for a lot of researches [1, 2] on the process of numerical (equivalent to – instrumental) computation as a physical process. But so far it remains unsolved problem: If numerical computation is a physical process, so how in this case to estimate the error generated by the physical process of computation [7]?

At least, the cause of this error can be detected by the physical analysis of numerical (instrumental) computation of derivative. For example, if $\boldsymbol{f}$ and $\boldsymbol{t}$ is the pair of measured (or instrumental computing) parameters and the uncertainties $\Delta\boldsymbol{f}$ and $\Delta\boldsymbol{t}$ of theirs natural measurements or computations are related by uncertainty relation:

$$\boldsymbol{\Delta f \cdot \Delta t \geq 1},$$

their difference ratio

$$\boldsymbol{\Delta f / \Delta t \ \geq 1/(\Delta t)^2},$$

converged to a limit, leads necessarily to the divergence

$$\lim_{\boldsymbol{\Delta t \to 0}} (\boldsymbol{\Delta f / \Delta t}) \to \infty.$$

Obviously, the resulting divergence occurring regardless of the kind of the dependence $\boldsymbol{f = f(t)}$ is a consequence of the fact that the values $\boldsymbol{\Delta f}$ and $\boldsymbol{\Delta t}$ being subject of the physical relation $\boldsymbol{\Delta f \cdot \Delta t \geq 1}$ cannot be reduced together. This fact points to a previously unconsidered physical source of incorrectness of numerical differentiation [3, 4].

Uncertainty relation in its classical form does not contain dynamic parameters of physical process. As will be shown below, the energy and dynamics as attributes of a physical process just are sources of uncertainty in measurements and computations. This is the reason for the inconsistency of analysis of instrumental computing as a physical process based on traditional forms of uncertainty relation.

Search on articles indicates that the uncertainty principle of the physics does not impose restrictions on the fundamental process of computation. Authors of the articles expressed disappointment at unsuitability of traditional UR of the physics on fundamental limitations of instrumental computation process.

Many explanations of incorrectness of numerical differentiation given in scope of mathematics are cut off from the root causes of the physical process of the instrumental computation.

N. S. Bakhvalov [8] summarizes the problem and writes "... the existence of a large number of different ways to approach is the lack of a simple explanation to the formulation and solution of the problem".

Adjectives

An interval in interval analysis is extension of concept of a real number; it is closed numerical space [5].

An interval uncertainty in interval analysis is incomplete knowledge about value which we are interested, when we can only specify its membership to this interval [6].

An optimal interval uncertainty is a key element of this development, is one of two multipliers of the Dynamic UR, and is a local quantum [3, 7]. Its length determined at each step. It has a function of a ***computing element*** and ***extrapolation step***. It provides minimum uncertainty of natural measurement or instrumental computing [7, 9]. It degenerates to a real number when $\boldsymbol{\mu} \to \infty$.

Total error is the sum of static (known) component and dynamic (obtained latter) component [3, 9]:

$$\Delta \boldsymbol{f} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{\mu}} \cdot \Delta \boldsymbol{t}} + \frac{\boldsymbol{f}'(\boldsymbol{t}) \cdot \Delta \boldsymbol{t}}{\mathbf{2}}.$$

An instrumental computation – equivalent to numerical calculation via computer.

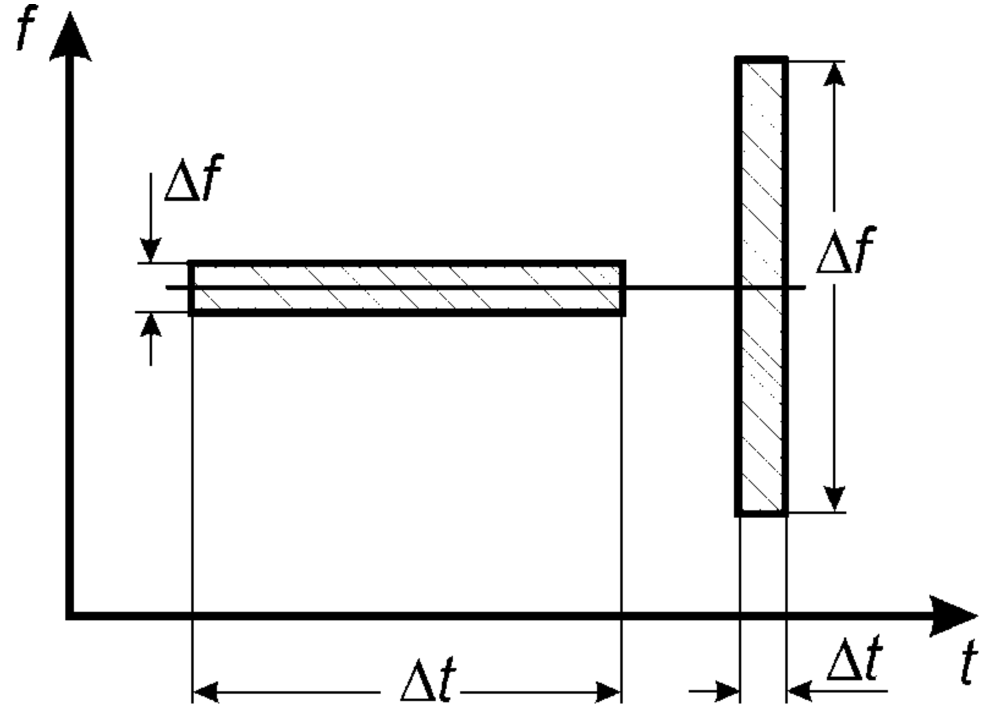
Parameter μ is the ratio of determinate component to stochastic component of physical value. This parameter is an attributive parameter of radiophysics [10]. In natural measurements and numerical computations this parameter gets value $0 < \mu < \infty$.

Natural measurement as a physical process contains energy.

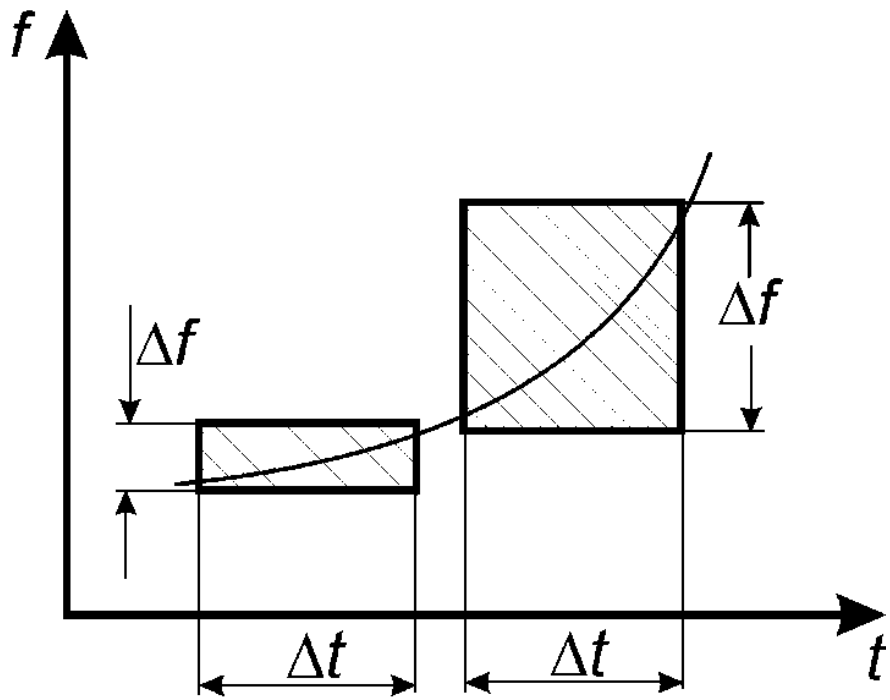
Numerical computation as a physical process contains energy [1].

Postulate and problem definition

1. Statics:



2. Dynamics:



1. Statics:

$$\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{t} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\mu}}} = \mathbf{const}$$

$$\Delta \boldsymbol{f} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\mu}} \cdot \Delta \boldsymbol{t}}$$

2. Dynamics:

$$\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{t} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{\mu}}} + \frac{\boldsymbol{f}'(\boldsymbol{t}) \cdot \Delta \boldsymbol{t}^{\mathbf{2}}}{\mathbf{2}} \neq \mathbf{const}$$

Introduction of dynamics in UR of classical (macroscopic) physics requires a new fundamental approach. The proposed approach represents a postulate of generalization of the macroscopic uncertainty relations of the physics. The proposed postulate of dynamic uncertainty relation (***Dynamic UR***) is a basis of problem definition in this work.

Problem solution

$$\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{t} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{\mu}}} + \frac{\boldsymbol{f}'(\boldsymbol{t}) \cdot \Delta \boldsymbol{t}^{\mathbf{2}}}{\mathbf{2}}$$

– postulate: dynamic generalization of uncertainty relation

$$\Delta \boldsymbol{f} = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{\mu}} \cdot \Delta \boldsymbol{t}} + \frac{\boldsymbol{f}'(\boldsymbol{t}) \cdot \Delta \boldsymbol{t}}{\mathbf{2}}$$

– total error

The kind of dependence of the total error on the variable $\Delta \boldsymbol{t}$ allows to minimize the objective function $\Delta \boldsymbol{f}$ which represents the measurement inaccuracy of frequency, for example. It allows us to find the optimal time interval $\Delta \boldsymbol{t}^{*}$ and the corresponding minimal uncertainty interval $\Delta \boldsymbol{f}_{\boldsymbol{min}}$.

Multiplication of these intervals converts dynamic form of UR into the ***interval*** dynamic form of UR [3, 9].

Dynamic *interval* uncertainty relation is:

$$\Delta \boldsymbol{f}_{\boldsymbol{min}\ \boldsymbol{i}} \cdot \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\mu}}}\,.$$

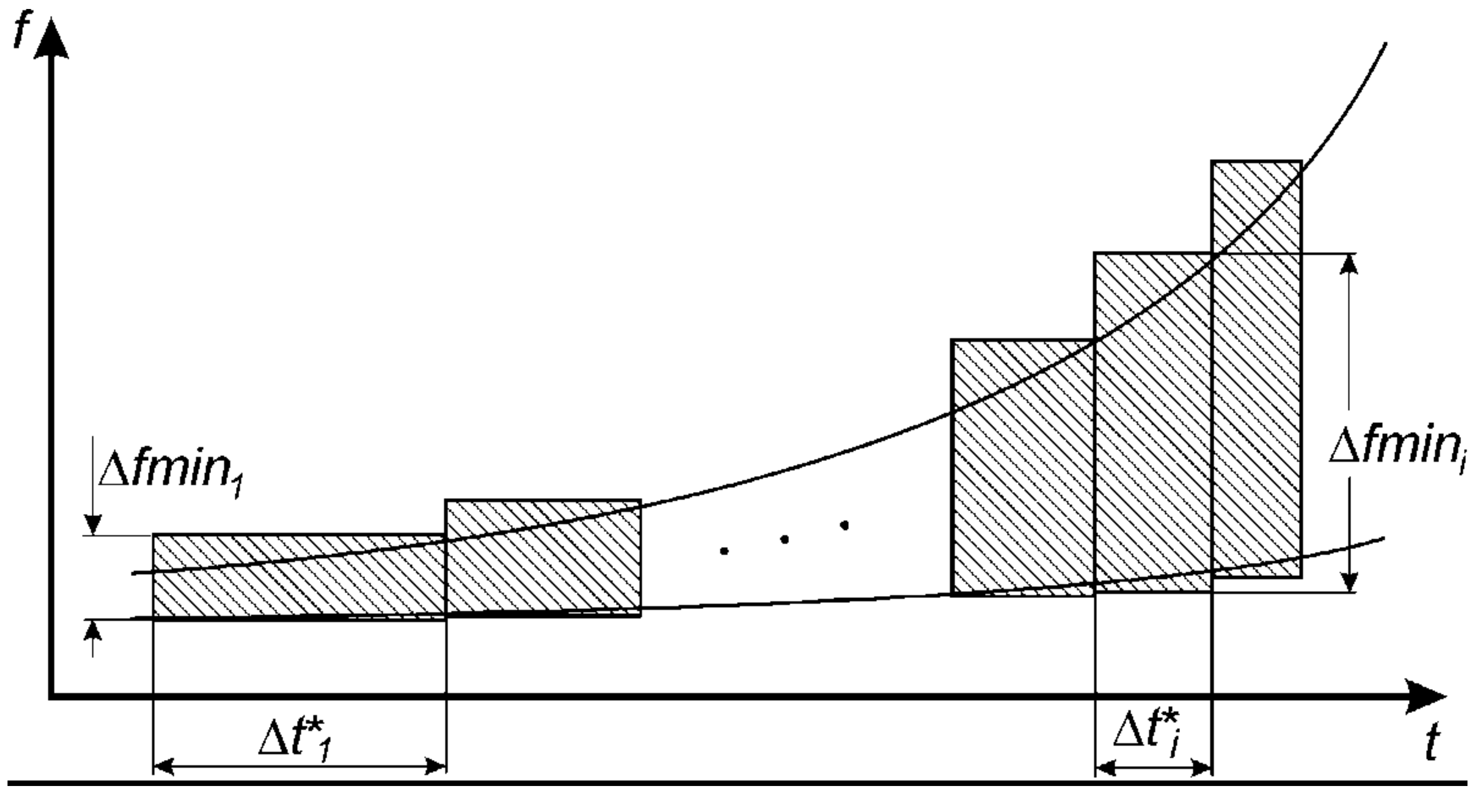
Solutions of
dynamic interval UR
are following:

$$\boldsymbol{\Delta t_i^* = 1 / \left(\sqrt{\mu_{i-1}} \cdot \left| \frac{\Delta f_{min}(\Delta t_{i-1}^*)}{\Delta t_{i-1}^*} \right| \right)^{0,5}}$$ – optimal uncertainty interval (1)

$$\boldsymbol{t_{i+1} = t_i + \beta} \cdot \Delta \boldsymbol{t_i^*}(\boldsymbol{\mu_{i-1}}, \boldsymbol{t_{i-1}})$$ – adapted dynamical grid, $\beta > 0$ (2)

$$\Delta \boldsymbol{f_{min}}(\Delta \boldsymbol{t_i^*}) = \left(\frac{\mathbf{1}}{\sqrt{\boldsymbol{\mu_{i-1}}}} \left| \frac{\Delta \boldsymbol{f_{min}}(\Delta \boldsymbol{t_{i-1}^*})}{\Delta \boldsymbol{t_{i-1}^*}} \right| \right)^{\mathbf{0,5}}$$ – interval of uncertainty minimum (3)

Physics of measurement and computation: The “cutting” of non-stationary process into optimal uncertainty intervals.



Measurement and computation within the optimal interval provide the minimum energy of measurement and computation as well as the minimum errors [3, 7, 9].

<u>The result of numerical (instrumental, natural) differentiation</u>: Difference derivative (natural derivative) $\boldsymbol{F}^{\angle}\left(\Delta \boldsymbol{f}_{\boldsymbol{min}(\boldsymbol{i})}, \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*}\right)$ calculated within sample units via algorithm (1) – (3) (notation $\boldsymbol{F}^{\angle}$ introduced to use by S. P. Shary):

$$\boldsymbol{F}^{\angle}\left(\Delta \boldsymbol{f}_{\boldsymbol{min}(\boldsymbol{i})}, \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*}\right) = \frac{\Delta \boldsymbol{f}_{\boldsymbol{min}(\boldsymbol{i})}}{\boldsymbol{\beta} \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*}} = \frac{\boldsymbol{f}(\boldsymbol{t}_{\boldsymbol{i}}) - \boldsymbol{f}(\boldsymbol{t}_{\boldsymbol{i-1}})}{\boldsymbol{t}_{\boldsymbol{i}} - \boldsymbol{t}_{\boldsymbol{i-1}}}; \tag{4}$$

where

$$\boldsymbol{t}_{\boldsymbol{i}} - \boldsymbol{t}_{\boldsymbol{i-1}} = \boldsymbol{\beta} \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*};$$

$$\boldsymbol{f}(\boldsymbol{t}_{\boldsymbol{i}}) - \boldsymbol{f}(\boldsymbol{t}_{\boldsymbol{i-1}}) = \Delta \boldsymbol{f}_{\boldsymbol{min}(\boldsymbol{i})};$$

$$\Delta \boldsymbol{f}_{\boldsymbol{min}}(\Delta \boldsymbol{t}^{*}) \cdot \Delta \boldsymbol{t}^{*}\left(\boldsymbol{\mu}, \boldsymbol{F}\left(\Delta \boldsymbol{f}_{\boldsymbol{min}(\boldsymbol{i})}, \Delta \boldsymbol{t}_{\boldsymbol{i}}^{*}\right)\right) = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\mu}}}. \tag{5}$$

- Difference derivative (4) is free of numerical differentiation error (it is free of incorrectness).
- Ratios (4), (5) show: if $\boldsymbol{\mu} \to \infty$ then the natural derivative degenerates into artifact – the classical derivative [4].

Experimental confirmation of Dynamic interval UR postulate

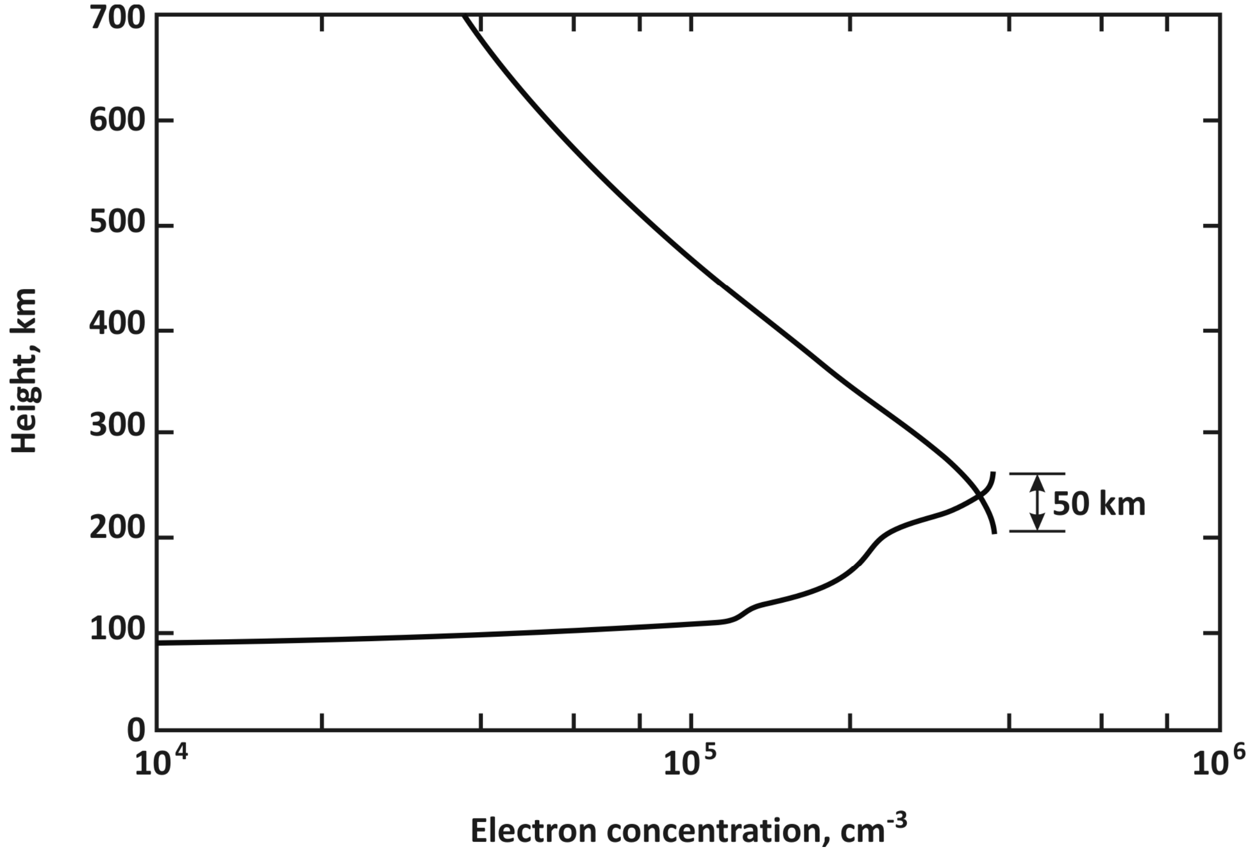


Fig. 5.5. [11] The distribution of electron concentration, obtained from the radio sounding of the ionospheric layers. This picture was obtained from the data of the external (from the satellite) and ground (from the bottom) radio sounding of the ionospheric layers.

Instead of ***the expected convergence*** of the ionograms at point N_emax ***we see them divergence*** in the neighborhood of this point. Explanation of this effect was not known [11]. ***The reason is: the natural interval of Dynamic interval UR cannot be equal to zero***.

The new approach to measuring process gives explaining for this unremovable divergence.

This unremovable divergence is an experimental confirmation of the Dynamic interval UR postulate.

Numerical tests (Numerical confirmation of Dynamic UR)

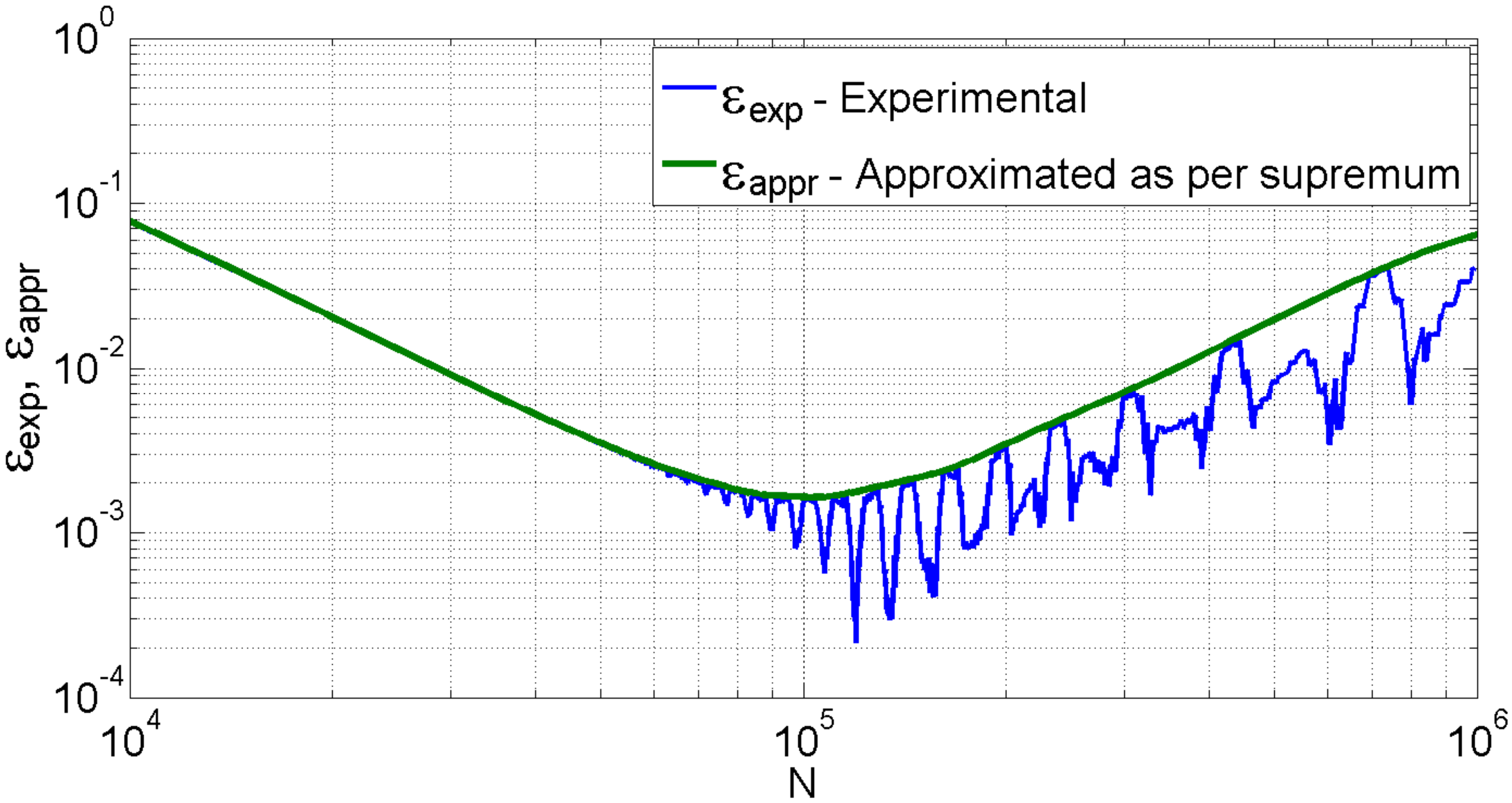
Numerical tests with Dynamic UR performed at four applications:

1. Numerical integration by trapezoidal
2. Solving Cauchy problem by Euler method 1^{st} order
3. Solving Cauchy problem by Euler method 2^{nd} order
4. Solving Cauchy problem by Runge-Kutta method 4^{th} order

When solving the numerical problems the range of the argument was broken up into an integer number of discrete steps. The length of the steps was determined using two approaches:

- The first (standard) case: the steps have constant length. The number of steps and their length are varied parameters.
- The second (proposed) case: the steps chosen via adaptive method in accordance with the Dynamic UR. At the first step (sample unit) the optimal interval determined in few iterations. At each following step (sample unit) the optimal interval determined using the previous interval. Step length assigned with optimal interval multiplied by the varied coefficient $\boldsymbol{\beta}$.

In each example under consideration we have implemented the series of tests with slowly increasing parameter $\boldsymbol{N}$ (number of steps). There were received experimental relations $\varepsilon_{\text{exp}}(\boldsymbol{N})$ and relations approximated as per supremum $\varepsilon_{\text{appr}}(\boldsymbol{N})$.



The first application: Numerical integration

Let us consider the function $\boldsymbol{y(x) = 1/x}$ and its integral

$$\boldsymbol{I = \int_{e^{-p}}^{1} \frac{1}{x}\, dx = p.}$$

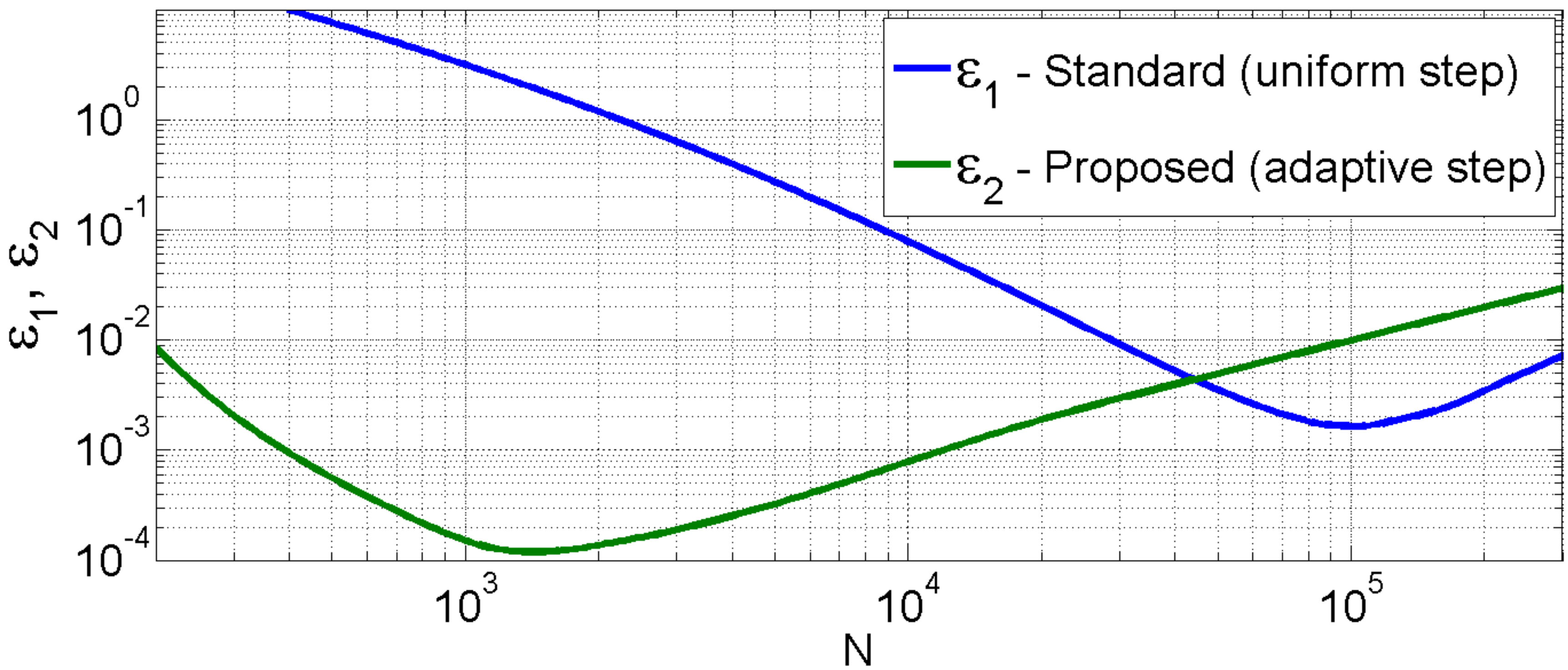
Criterion for the accuracy of the method was the value of the accumulated error over the area of integration, obtained by comparing the calculated results with the analytically found exact value of the integral:

$$\boldsymbol{\varepsilon(N) = |I(N) - p|.}$$

It was used single precision (float), i.e. bit capacity of computer had order $\boldsymbol{c \approx 10^7}$.

We have obtained dependencies of accuracy from the number of steps:

- $\boldsymbol{\varepsilon_1 = \varepsilon_1(N)}$ – In standard algorithm (with uniform steps).
- $\boldsymbol{\varepsilon_2 = \varepsilon_2(N)}$ – In proposed algorithm (adaptive method according to Dynamic UR).



Standard algorithm		Adaptive algorithm		Advantage
Steps	Error	Steps	Error	
104700	$1.64 \cdot 10^{-3}$	1400	$1.19 \cdot 10^{-4}$	**Steps reduced in 74 times.** **Error reduced in 13.8 times.**

The second and third applications: Solving Cauchy problem by Euler method 1st and 2nd order

Let us considered the function $\boldsymbol{y(x) = 1/x}$.

The problem is to find numerically $\boldsymbol{y_N}$ at the sample unit $\boldsymbol{x_N = 1}$, if the function derivative is

$$\boldsymbol{y' = -y^2}$$

and initial conditions are

$$\boldsymbol{x_0 = 0.001}, \quad \boldsymbol{y(x_0) = 1/x_0}.$$

Criterion for the accuracy of the method was the value of error at the last sample unit:

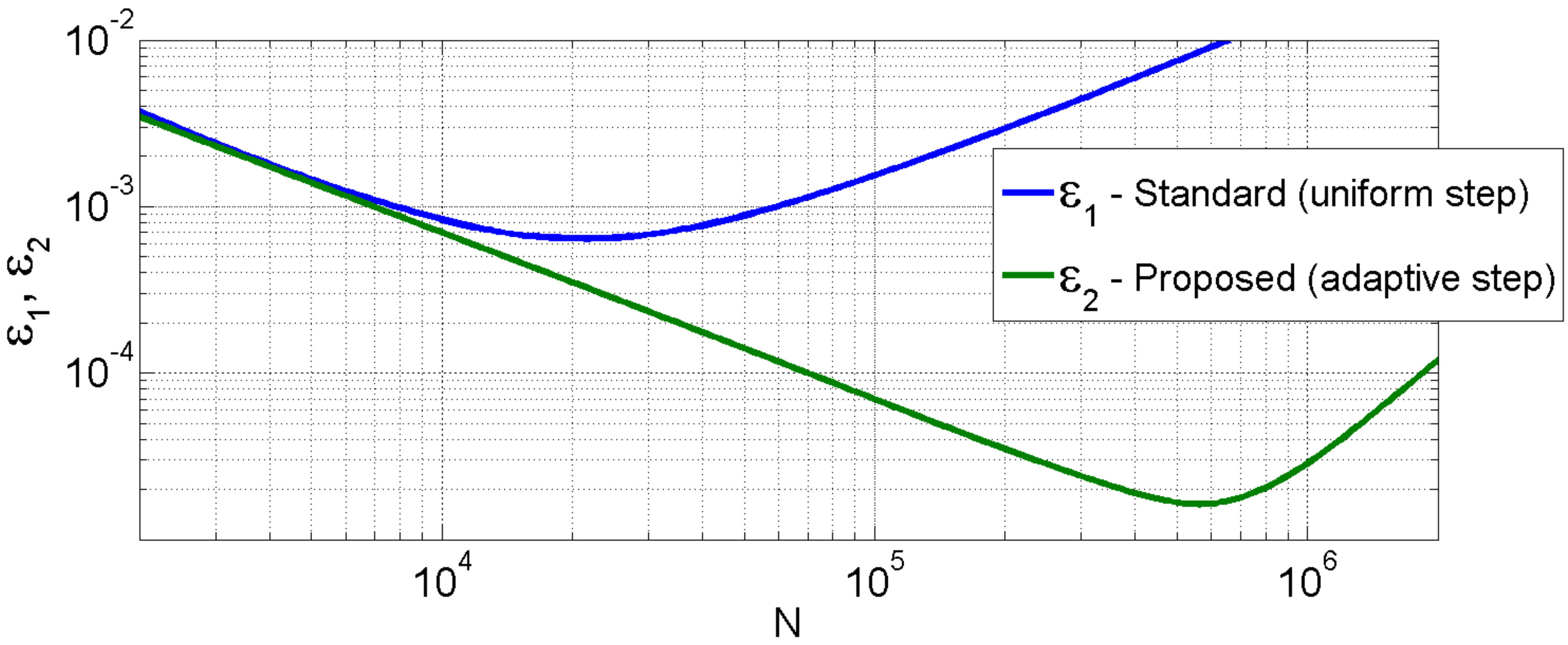
$$\boldsymbol{\varepsilon(N) = |y_N - y(x_N)|}.$$

It was used single precision (float), i.e. bit capacity of computer had order $\boldsymbol{c \approx 10^7}$.

We have obtained dependencies of accuracy from the number of steps:

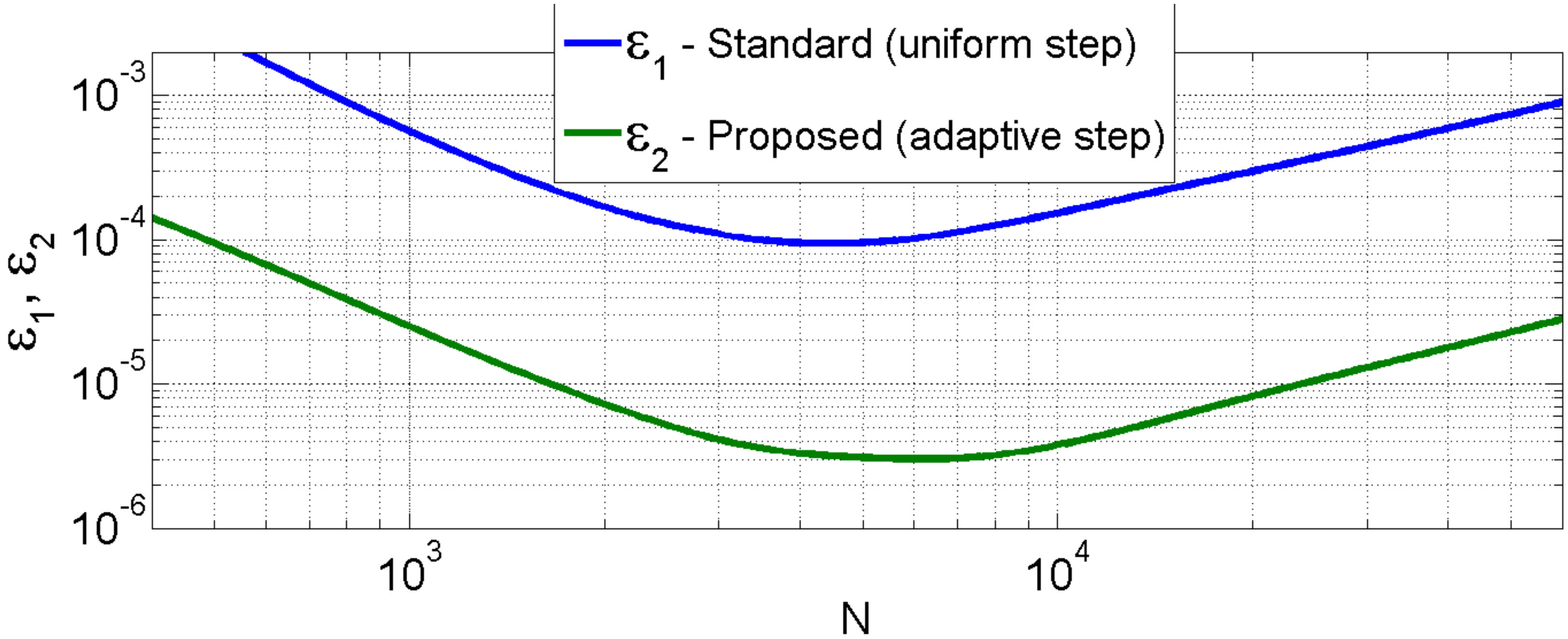
- $\boldsymbol{\varepsilon_1 = \varepsilon_1(N)}$ – In standard algorithm (with uniform steps).
- $\boldsymbol{\varepsilon_2 = \varepsilon_2(N)}$ – In proposed algorithm (adaptive method according to Dynamic UR).

Euler method, 1st order



Standard algorithm		Adaptive algorithm		Advantage
Steps	Error	Steps	Error	
104700	$1.64 \cdot 10^{-3}$	1400	$1.19 \cdot 10^{-4}$	Steps increased in 26 times. **Error reduced in 39 times.**

Euler method, 2nd order



Standard algorithm		Adaptive algorithm		Advantage
Steps	Error	Steps	Error	
4169	$9.44\cdot10^{-5}$	6000	$3.02\cdot10^{-6}$	Steps increased in 1.44 times. **Error reduced in 31 times.**

The fourth application: Solving Cauchy problem by Runge-Kutta method 4th order

Let us consider the function $\boldsymbol{y(x) = 1/x}$.

The problem is to find numerically $\boldsymbol{y_N}$ at the sample unit $\boldsymbol{x_N = 1}$, if the function derivative is

$$\boldsymbol{y' = -y^2}$$

and initial conditions are

$$\boldsymbol{x_0 = 0.001}, \quad \boldsymbol{y(x_0) = 1/x_0}.$$

Criterion for the accuracy of the method was the value of error at the last sample unit:

$$\boldsymbol{\varepsilon(N) = |y_N - y(x_N)|}.$$

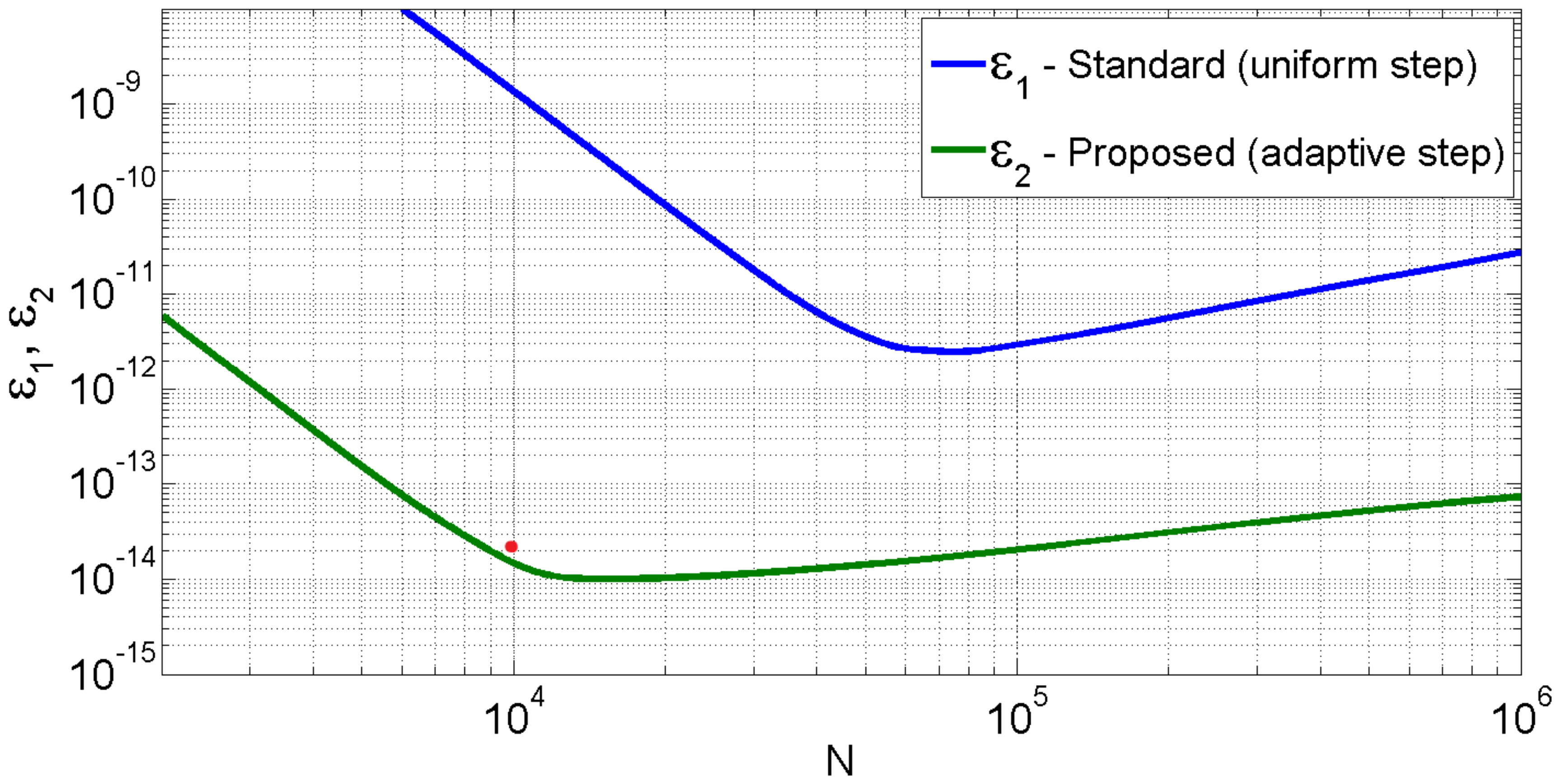
It was used double precision (double), i.e. bit capacity of computer had order $\boldsymbol{c \approx 10^{15}}$.

We examined three methods to obtain step length:

- First standard algorithm (uniform step).
  We have obtained the dependency of accuracy from the number of steps:
  $\boldsymbol{\varepsilon_1 = \varepsilon_1(N)}$.

- Second standard algorithm (step obtained adaptively via dichotomy in several iterations at the each sample unit). The step length reduced until the partial error estimated with Runge rule stops decreasing.
  We have obtained the accuracy $\boldsymbol{\varepsilon_2}$ and the steps number $\boldsymbol{N_2}$.

- Proposed algorithm (adaptive method according to the Dynamic UR).
  We have obtained the dependency of accuracy from the number of steps:
  $\boldsymbol{\varepsilon_3 = \varepsilon_3(N)}$.



Red point – result of 2nd standard (iteration) algorithm

Comparison of proposed method and first standard (uniform) method

1st standard algorithm		Adaptive algorithm		Advantage
Steps	Error	Steps	Error	
70630	$2.50\cdot10^{-12}$	13840	$1.01\cdot10^{-14}$	**Steps reduced in 5.1 times. Error reduced in 247 times.**

Comparison of proposed method and second standard (iteration) method

2nd standard algorithm		Adaptive algorithm		Advantage
Steps	Error	Steps	Error	
9880	$2.20\cdot10^{-14}$	13840	$1.01\cdot10^{-14}$	**Steps reduced in 1.4 times. Error reduced in 2.18 times.**

Discussion

1. Quantized structure of the dynamic uncertainty relation indicates that the interval character of a physical process (not always seen) is primary, not only in the micro, but also in the macrocosm.

2. The proposed algorithm inherits the difficult and complex character of a natural physical process. Nevertheless, the complexity of measuring and computational algorithms is compensated not only by resolving the problem of incorrectness of numerical differentiation, but also by solution of the fundamental problem of reducing the potential minimum uncertainties in instrumental computations and natural measurements.

3. The most important tool – the derivative – becomes even more important in its difference form. New difference derivative is an indispensable link in numerical computing. Thus, even in the numerical integration performed according to adaptive algorithm, accumulated over the entire area of integration error is minimal in comparison with known methods.

4. The interval of uncertainty owes its birth by a computer and interval analysis [5, 6, 12] originally was perceived as an artifact. However, the ***optimal*** interval of uncertainty is not an artifact. On the contrary, a real number can be considered as an artifact.

5. Grid function is primary, if its nodes placed at the extremities of optimal intervals.

6. Continuous function is secondary as derived from the nodes of the grid function or from the natural samples using several interpolation values.

7. The optimal interval is a measuring and computing element. Its value determined locally, at every step of computing or the measuring process. The optimal interval is a structureless quantum.

8. One-dimensional adaptive dynamical grid generated automatically by the optimal intervals.

9. On the order of the method. Runge rule for inaccuracy estimation in numerical methods does not take into account round-off component. So Runge rule is correct only for steps with length $\boldsymbol{h} \geq \boldsymbol{h}_{\boldsymbol{min}}$. Value $\boldsymbol{h}_{\boldsymbol{min}}$ corresponds to the minimum of the total error.

On practical application

1. The dynamic uncertainty relation is subject to any pair of interdependent data of natural measurement.

2. The dynamic uncertainty relation is also subject to any pair of instrumentally computing interdependent variables. For example, it may be uncertainty $\Delta\boldsymbol{y}$ of a function of a real variable and uncertainty of its argument $\Delta\boldsymbol{x}$.

3. In practice, if the function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ of a real argument is integrable in the square, then the uncertainties $\Delta\boldsymbol{y}$ and $\Delta\boldsymbol{x}$ are subordinated to dynamic interval uncertainty relation $\Delta\boldsymbol{y}_{\boldsymbol{min}}\Delta\boldsymbol{x}^{*} = \mathbf{1}/\sqrt{\boldsymbol{\mu}}$.

4. Sizeable (by orders of magnitude) decreasing of the number of nodes in the measurement and calculation allows the monitoring of fast processes in real time.

Conclusions

1. Natural measurements and instrumental computations have structurally identical algorithms.

2. The developed algorithm provides minimum uncertainty of natural measurements and instrumental evaluation.

3. The dynamic UR reveals the primacy of interval processes, not only in the world of micro parameters (known), but also in the world of macro parameters.

4. The optimal interval as the extrapolation step is the operational forecast with the least risk.

5. The developed adaptive algorithm is the tool of theoretical development on ADC of a new generation.

Appendix

On the history development of the concept of the uncertainty relation of the physics

Uncertainty relation for the frequency band Δf and duration Δt of the wave packet in the form Δf · Δt ≥ 1 was known in optics and acoustics from the XIX century.

$$\Delta f \cdot \Delta t \geq 1 \qquad \text{(XIX century)}$$

$$\Delta f \cdot \Delta t \geq \frac{1}{\sqrt{\mu}} \qquad \text{(XX century)}$$

$$\Delta f \cdot \Delta t = \frac{1}{2\sqrt{\mu}} + \frac{f(t)' \cdot \Delta t^2}{2} \qquad \text{(1992 year)}$$

$$\Delta f_{\min(i)} \cdot \Delta t_i^* = \frac{1}{\sqrt{\mu}}$$

Thank You
for Your attention

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