# Mapped Regular Pavings

Jennifer Harlow[†], **Raaz**esh Sainudiin[†] and Warwick Tucker[⋆]

[†]Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand
[⋆]Department of Mathematics, Uppsala University, Uppsala, Sweden

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   *Mapped Regular Pavings (MRPs)*
4. – **by** exploiting the *algebraic structure of partitions formed by finite-rooted-binary (frb) trees*
5. – **thereby** provide algorithms for several *inclusion algebras over frb tree partitions*

# arithmetic from intervals to their frb-tree partitions

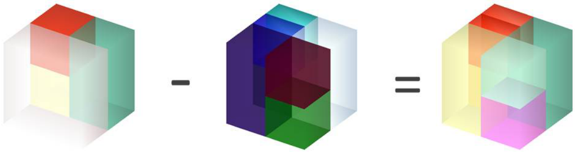

Figure: Arithmetic with coloured spaces.

# arithmetic from intervals to their frb-tree partitions

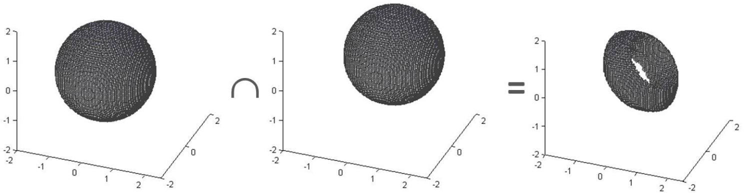


Figure: Intersection of enclosures of two hollow spheres.

# arithmetic from intervals to their frb-tree partitions

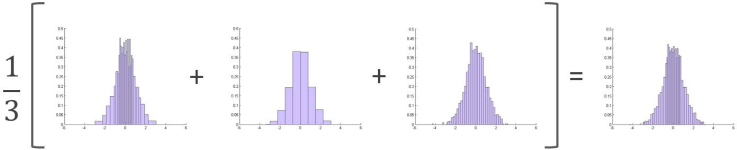

Figure: Histogram averaging.

# Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

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1. Arithmetic on piece-wise constant functions and interval-valued functions;
2. Exploiting the tree-based structure to obtain interval enclosures of real-valued functions efficiently
3. Statistical set-processing operations like marginal density, conditional density and highest coverage regions, visualization, etc

# An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

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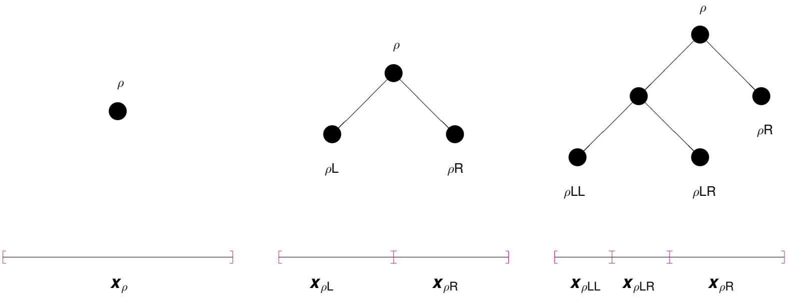
Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^1$



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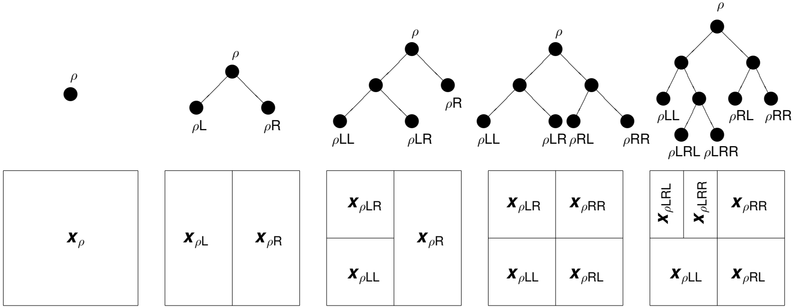

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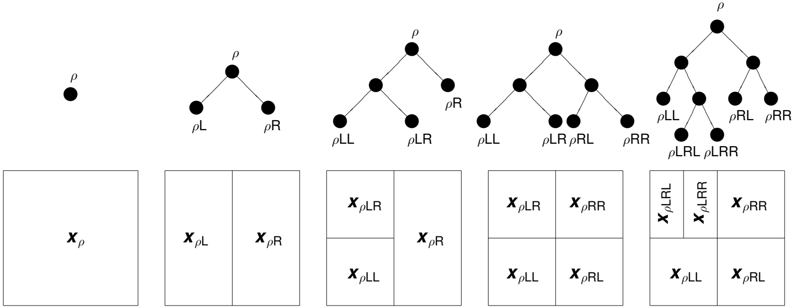
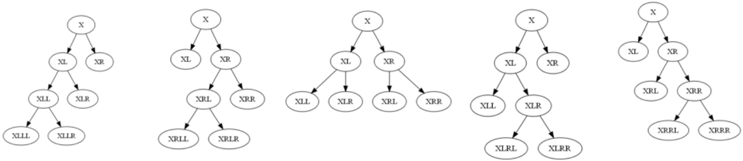



By this “RP Peano’s curve” frb-trees encode paritions of $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

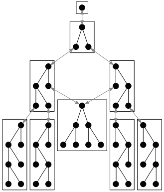
# Algebraic Structure and Combinatorics of RPs

## Leaf-depth encoded RPs

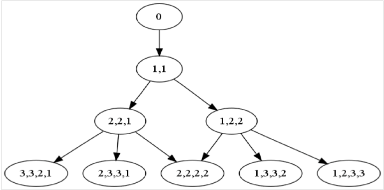



(3, 3, 2, 1) (1, 3, 3, 2) (2, 2, 2, 2) (2, 3, 3, 1) (1, 2, 3, 3)

There are $C_k$ RPs with $k$ splits

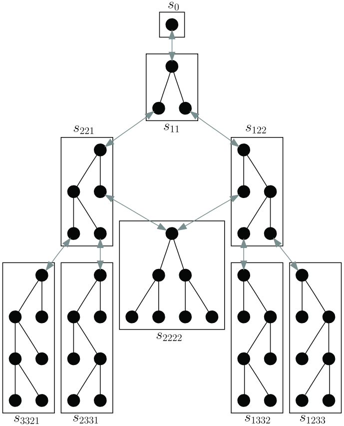


$$
\begin{aligned}
C_0 &= 1\\
C_1 &= 1\\
C_2 &= 2\\
C_3 &= 5\\
C_4 &= 14\\
C_5 &= 42\\
\ldots &= \ldots\\
C_k &= \frac{(2k)!}{(k+1)!k!}\\
\ldots &= \ldots\\
C_{15} &= 9694845\\
\ldots &= \ldots\\
C_{20} &= 6564120420\\
\ldots &= \ldots
\end{aligned}
$$



# Hasse (transition) Diagram of Regular Pavings

Transition diagram over $\mathbb{S}_{0:3}$ with split/reunion operations




RS, W.Taylor and G.Teng, Catalan Coefficients, Sequence A185155 in The On-Line Encyclopedia of Integer Sequences, 2012, `http://oeis.org`

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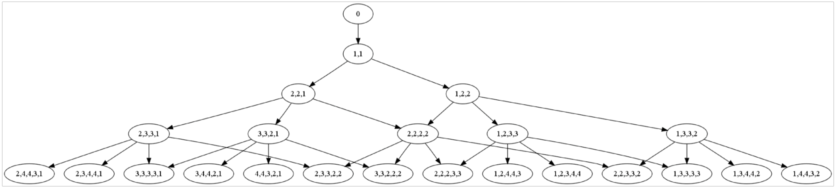
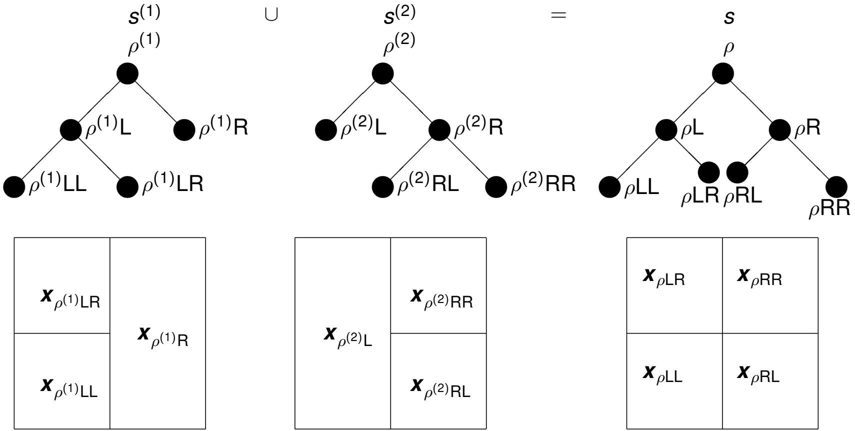
Transition diagram over $\mathbb{S}_{0:4}$ with split/reunion operations




1. The above state space is denoted by $\mathbb{S}_{0:4}$
2. Number of RPs with $k$ splits is the Catalan number $C_k$
3. There is more than one way to reach a RP by $k$ splits
4. Randomized enclosure algorithms are Markov chains on $\mathbb{S}_{0:\infty}$

# RPs are closed under union operations

$s^{(1)} \cup s^{(2)} = s$ is union of two RPs $s^{(1)}$ and $s^{(2)}$ of $\boldsymbol{x}_\rho \in \mathbb{R}^2$.



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**Lemma 1:** The algebraic structure of frb-trees (underlying Thompson's group) is closed under union operations.

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**Proof:** by a "transparency overlay process" argument (cf. Meier 2008).

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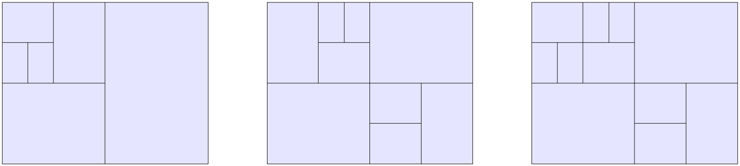

**Algorithm 1:** `RPUnion`$(\rho^{(1)}, \rho^{(2)})$

**input** : Root nodes $\rho^{(1)}$ and $\rho^{(2)}$ of RPs $s^{(1)}$ and $s^{(2)}$, respectively, with root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$

**output** : Root node $\rho$ of RP $s = s^{(1)} \cup s^{(2)}$

**if** `IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
  $\rho \leftarrow$ `Copy`$(\rho^{(1)})$
  **return** $\rho$
**end**

**else if** !`IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
  $\rho \leftarrow$ `Copy`$(\rho^{(1)})$
  **return** $\rho$
**end**

**else if** `IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$ **then**
  $\rho \leftarrow$ `Copy`$(\rho^{(2)})$
  **return** $\rho$
**end**

**else**
  !`IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$
**end**
Make $\rho$ as a node with $\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$
Graft onto $\rho$ as left child the node `RPUnion`$(\rho^{(1)}\mathrm{L}, \rho^{(2)}\mathrm{L})$
Graft onto $\rho$ as right child the node `RPUnion`$(\rho^{(1)}\mathrm{R}, \rho^{(2)}\mathrm{R})$
**return** $\rho$

Note: this is not the minimal union of the (Boolean mapped) RPs of Jaulin et. al. 2001

# Dfn: Mapped Regular Paving (MRP)

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- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of $s$, respectively.

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- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of $s$ to an element in $\mathbb{Y}$ as follows:

$$\{\rho v \mapsto f_{\rho v} : \rho v \in \mathbb{V}(s), f_{\rho v} \in \mathbb{Y}\} \ .$$

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- Such a map $f$ is called a $\mathbb{Y}$*-mapped regular paving ($\mathbb{Y}$-MRP)*.

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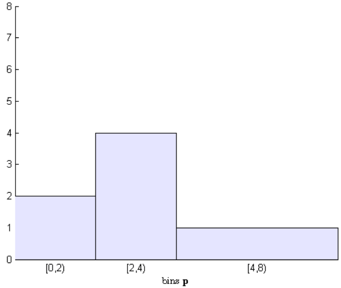
- Such a map $f$ is called a $\mathbb{Y}$*-mapped regular paving (*$\mathbb{Y}$*-MRP)*.
- Thus, a $\mathbb{Y}$-MRP $f$ is obtained by augmenting each node $\rho v$ of the RP tree $s$ with an additional data member $f_{\rho v}$.

# Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{R}$

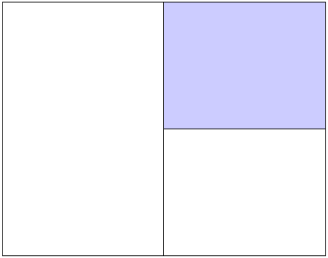
$\mathbb{R}$-MRP over $s_{221}$ with $x_\rho = [0, 8]$



# Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{B}$

$\mathbb{B}$-MRP over $s_{122}$ with $x_\rho = [0,1]^2$ (e.g. Jaulin et. al. 2001)

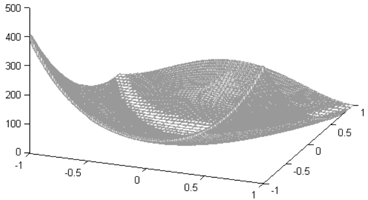


# Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{IR}$

– frb tree representation for interval inclusion algebra

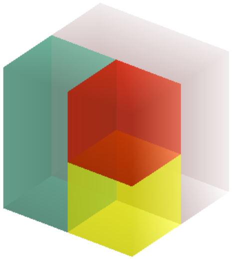
$\mathbb{IR}$-MRP enclosure of the Rosenbrock function with $x_\rho = [-1, 1]^2$



# Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = [0, 1]^3$
– R G B colour maps

$[0, 1]^3$-MRP over $s_{3321}$ with $x_\rho = [0, 1]^3$

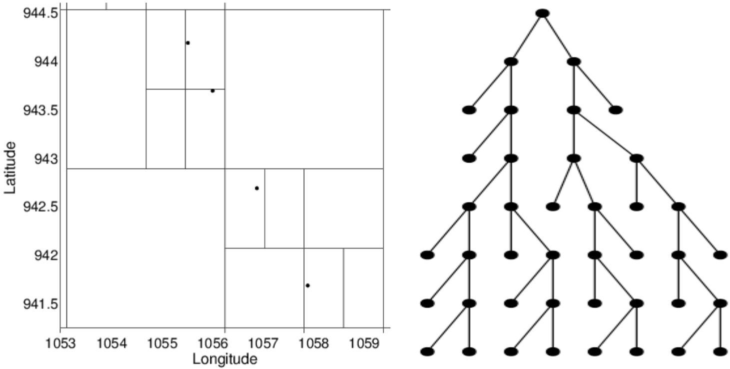


# Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{Z}_+ := \{0, 1, 2, ...\}$

– radar-measured aircraft trajectory data

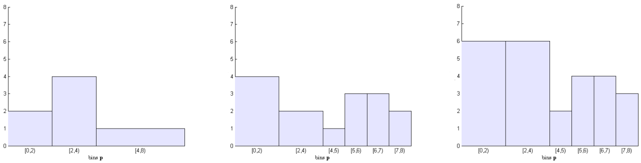
$\mathbb{Z}_+$-MRP trajectory of an aircraft and its tree



# $\mathbb{Y}$-MRP Arithmetic

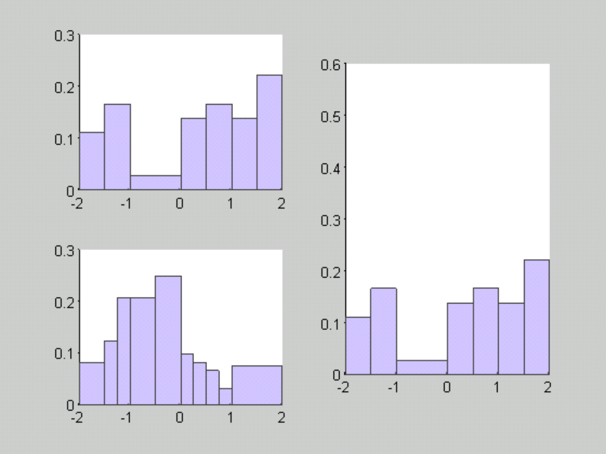
If $\star : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ then we can extend $\star$ point-wise to two $\mathbb{Y}$-MRPs $f$ and $g$ with root nodes $\rho^{(1)}$ and $\rho^{(2)}$ via `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$.
This is done using `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, +)$

$f$ $g$ $f+g$



# $\mathbb{R}$-MRP Addition by $\texttt{MRPOperate}(\rho^{(1)}, \rho^{(2)}, +)$

adding two piece-wise constant functions or $\mathbb{R}$-MRPs



## **Algorithm 2:** `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$

**input** : two root nodes $\rho^{(1)}$ and $\rho^{(2)}$ with same root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$ and binary operation $\star$.
**output** : the root node $\rho$ of $\mathbb{Y}$-MRP $h = f \star g$.

Make a new node $\rho$ with box and image
$\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$; $h_\rho \leftarrow f_{\rho^{(1)}} \star g_{\rho^{(2)}}$

**if** `IsLeaf`$(\rho^{(1)})$ & `!IsLeaf`$(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{L}}$; $\boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{R}}$
  $f_{\mathrm{L}'} \leftarrow f_{\rho^{(1)}}$, $f_{\mathrm{R}'} \leftarrow f_{\rho^{(1)}}$
  Graft onto $\rho$ as left child the node `MRPOperate`$(\mathrm{L}', \rho^{(2)}\mathrm{L}, \star)$
  Graft onto $\rho$ as right child the node `MRPOperate`$(\mathrm{R}', \rho^{(2)}\mathrm{R}, \star)$
**end**

**else if** `!IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{L}}$; $\boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{R}}$
  $g_{\mathrm{L}'} \leftarrow g_{\rho^{(2)}}$, $g_{\mathrm{R}'} \leftarrow g_{\rho^{(2)}}$
  Graft onto $\rho$ as left child the node `MRPOperate`$(\rho^{(1)}\mathrm{L}, \mathrm{L}', \star)$
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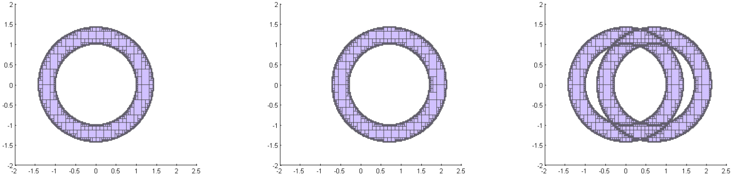
**end**

**else if** `!IsLeaf`$(\rho^{(1)})$ & `!IsLeaf`$(\rho^{(2)})$ **then**
  Graft onto $\rho$ as left child the node `MRPOperate`$(\rho^{(1)}\mathrm{L}, \rho^{(2)}\mathrm{L}, \star)$
  Graft onto $\rho$ as right child the node `MRPOperate`$(\rho^{(1)}\mathrm{R}, \rho^{(2)}\mathrm{R}, \star)$
**end**
**return** $\rho$

# $\mathbb{B}$-MRP arithmetic

Two Boolean-mapped regular pavings $A_1$ and $A_2$ and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

$A_1$ $\quad$ $A_2$ $\quad$ $A1 + A2$



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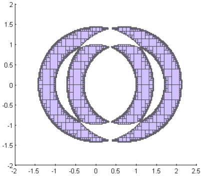
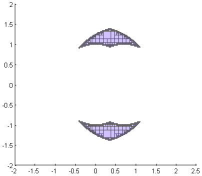
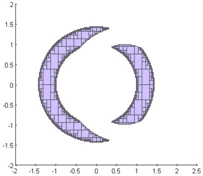
$A_1 - A2$




$A_1 \times A_2$



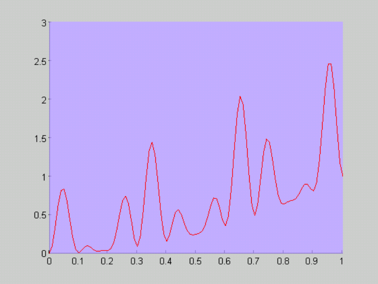

$A_1 \div A_2$



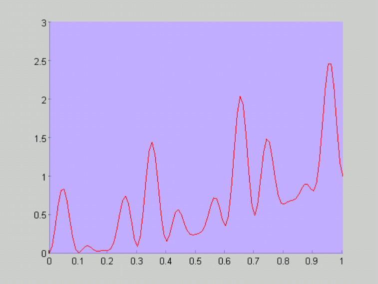
# Example – Prioritised Splitting

inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x} + 1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$
priority function: $\psi(\rho\mathrm{v}) = \mathrm{vol}\,(\rho\mathrm{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{v}}))$

To 50 leaves by
$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 50)$




To 100 leaves by
$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$



**Algorithm 3:** $\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell})$

**input** : $\rho$, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP $s$, root box $\boldsymbol{x}_\rho$ and $\boldsymbol{f}_\rho = \boldsymbol{g}(\boldsymbol{x}_\rho)$,
$\psi : \mathbb{L}(s) \rightarrow \mathbb{R}$ such that
$\psi(\rho \mathrm{v}) = \mathrm{vol}\,(\boldsymbol{x}_{\rho\mathrm{v}})\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{v}}) - 0.5\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{vL}}) + \boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{vR}}))),$
$\bar{\ell}$ the maximum number of leaves.

**output** : $\boldsymbol{f}$ with modified RP $s$ such that $|\mathbb{L}(s)| = \bar{\ell}$

**if** $|\mathbb{L}(s)| < \bar{\ell}$ **then**

$$\rho\mathrm{v} \leftarrow \texttt{random\_sample}\left(\operatorname*{argmax}_{\rho\mathrm{v} \in \mathbb{L}(s)} \psi(\rho\mathrm{v})\right)$$

Split $\rho$v: $\triangledown(\rho\mathrm{v}) = \{\rho\mathrm{vL}, \rho\mathrm{vR}\}$ `// split the sampled node`

$\boldsymbol{f}_{\rho\mathrm{vL}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathrm{vL}}))$

$\boldsymbol{f}_{\rho\mathrm{vR}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathrm{vL}}))$

$\texttt{RPQEnclose}^{\triangledown}(\rho, \psi, \bar{\ell})$

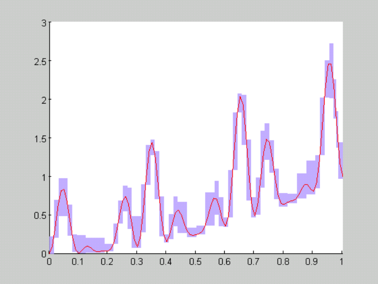
**end**

# Example - Prioritised Splitting Continued

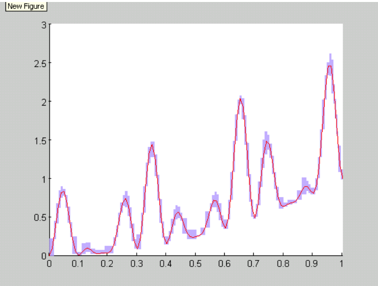
inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x}+1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$

priority function: $\psi(\rho\mathrm{v}) = \mathrm{vol}\,(\rho\mathrm{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{v}}))$

To 50 leaves by $\mathtt{RPQEnclose}^{\nabla}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 50)$




To 100 leaves by $\mathtt{RPQEnclose}^{\nabla}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$




Can we get tighter enclosures using only 50 leaves by propagating the interval hull of 100-leaved $\mathbb{IR}$-MRP up the tree and then doing a prioritised merging of the cherries?

# Hull Propagate up the tree via HullPropagate($\rho$)

**Algorithm 4:** HullPropagate($\rho$)

**input** : $\rho$, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP $s$.
**output** : Modify input MRP $\boldsymbol{f}$.

```
if !IsLeaf(ρ) then
    HullPropagate(ρL)
    HullPropagate(ρR)
    f_ρ ← f_ρL ⊔ f_ρR
end
```

By calling HullPropagate($\rho$) on our $\mathbb{IR}$-MRP of Example constructed by $\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$ we would have tightened the range enclosures of $\boldsymbol{g}$ in the internal nodes.

# Prioritised Merging via $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

**Algorithm 5:** $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

**input** : $\rho$, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP $s$, box $\boldsymbol{x}_{\rho}$,
$\psi : \mathbb{C}(s) \to \mathbb{R}$ as $\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\boldsymbol{x}_{\rho\mathsf{v}})\,(\boldsymbol{f}_{\rho\mathsf{v}} - 0.5\,(\boldsymbol{f}_{\rho\mathsf{vL}} + \boldsymbol{f}_{\rho\mathsf{vR}}))$,
$\bar{\ell}'$ the maximum number of leaves.

**output** : modified $\boldsymbol{f}$ with RP $s$ such that $|\mathbb{L}(s)| = \bar{\ell}'$ or $\mathbb{C}(s) = \emptyset$.

**if** $|\mathbb{L}(s)| \geq \bar{\ell}'$ & $\mathbb{C}(s) \neq \emptyset$ **then**

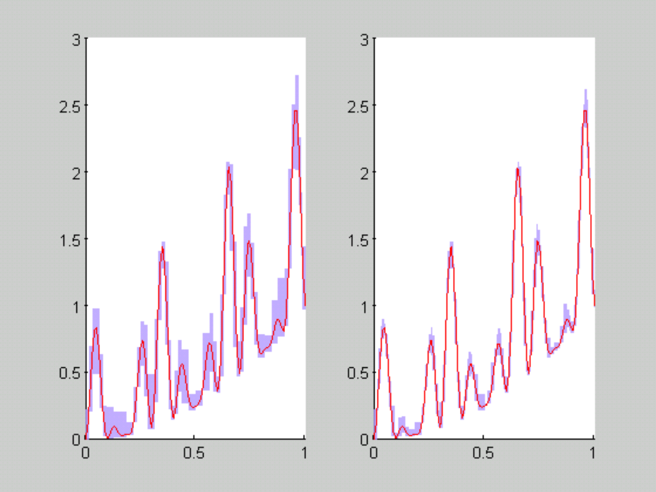
```
ρv ← random_sample (argmin_{ρv∈ℂ(s)} ψ(ρv))      // choose a
random node with smallest ψ
Prune(ρL)
Prune(ρR)
RPQEnclose△(ρ, ψ, ℓ̄′)
```

**end**

# Example – Split, Propogating & Prune

Yes we can!

$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$; $\texttt{HullPropagate}(\rho)$; $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}' = 50)$

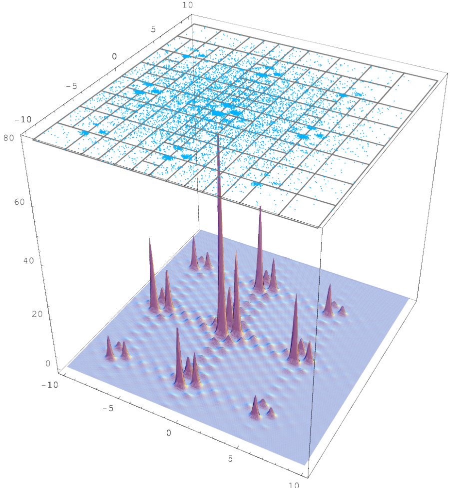


# Statistical Applications

- "Nonparametric Density Estimation" with massive metric data streams
- Stat. Operations: Coverage, Marginal integral and Slice
- Memory-efficient Arithmetic for Air Traffic Co-trajectories
- Life Science Appl.: Animal Migration Track
- Bold untried Idea: Set-valued Arithmetic for Geospatial Data (Global EQ data)

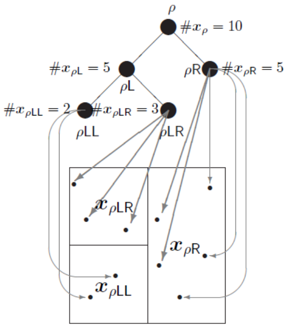
# Nonparametric Density Estimation

Problem: Take samples from an unknown density $f$ and consistently reconstruct $f$

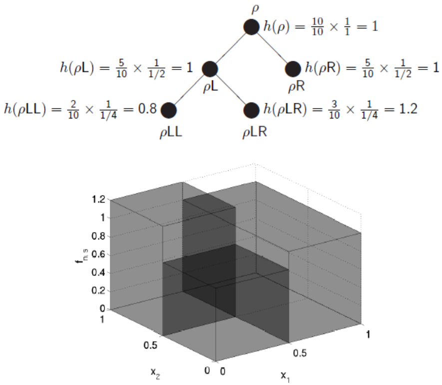


# Nonparametric Density Estimation

Approach: Use statistical regular paving to get $\mathbb{R}$-MRP data-adaptive histogram




(a) An SRP tree and its constituents.

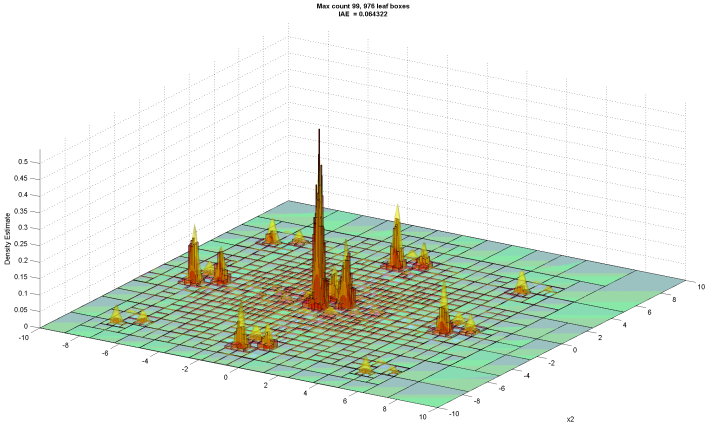



(b) An SRP histogram and its tree.

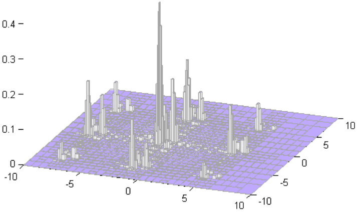
# Nonparametric Density Estimation

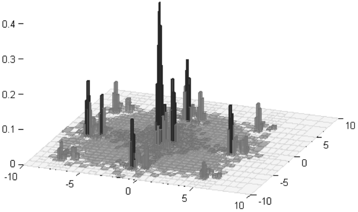
Solution: $\mathbb{R}$-MRP histogram averaging allows us to produce a consistent Bayesian estimate of the density (up to 10 dimensions)

(Teng, Harlow, Lee and S., *ACM Trans. Mod. & Comp. Sim.*, [r. 2] 2012)



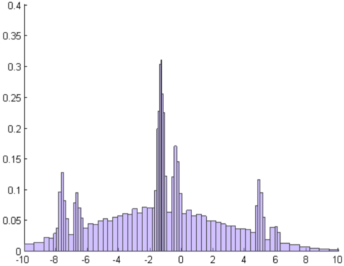
# Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP

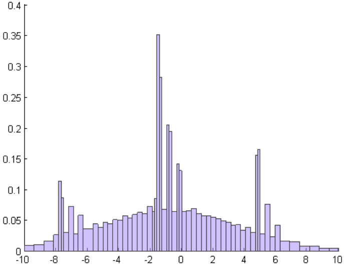



$\mathbb{R}$-MRP approximation to Levy density and its coverage regions with $\alpha = 0.9$ (light gray), $\alpha = 0.5$ (dark gray) and $\alpha = 0.1$ (black)

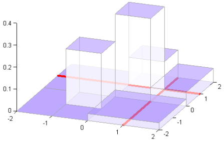
# Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP

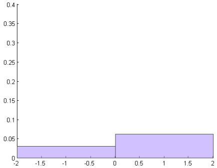


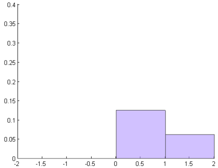




Marginal densities $f^{\{1\}}(x_1)$ and $f^{\{2\}}(x_2)$ along each coordinate of $\mathbb{R}$-MRP approximation

# Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP



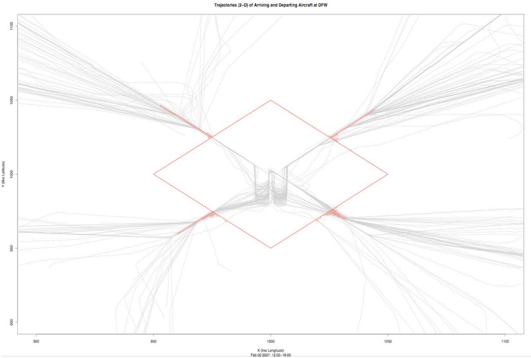




The slices of a simple $\mathbb{R}$-MRP in 2D

# Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

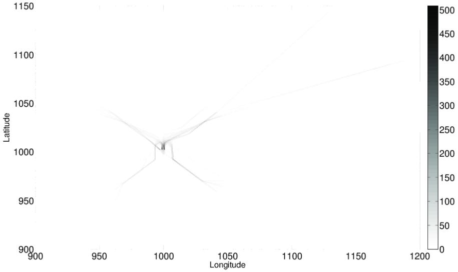
## On a Good Day



# Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

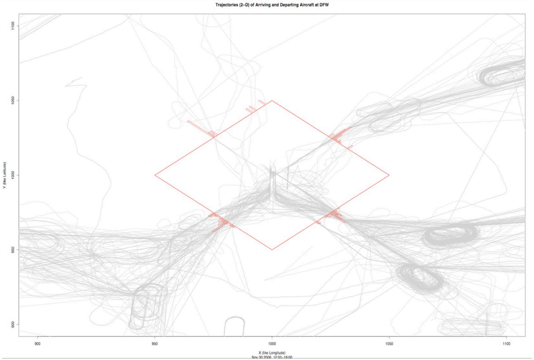
## $\mathbb{Z}_+$-MRP On a Good Day



# Air Traffic “Arithmetic” → dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

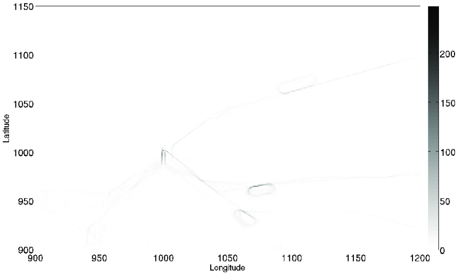
## On a Bad Day



# Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

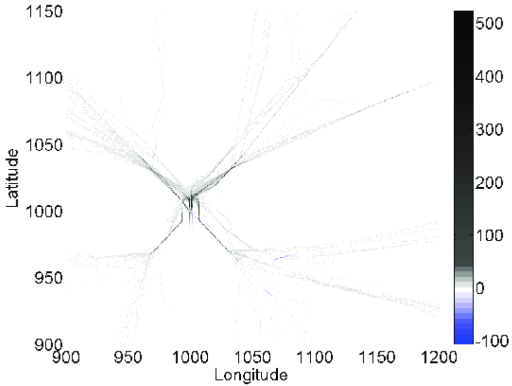
## $\mathbb{Z}_+$-MRP On a Bad Day



# Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

$\mathbb{Z}_+$-MRP pattern for Good Day $-$ Bad Day



# Conclusions

- $\mathbb{Y}$-MRPs provide frb-tree partition arithmetic
- $\mathbb{I}\mathbb{Y}$-MRPs allow efficient arithmetic for Neumaier's inclusion algebras
- $\mathbb{I}\mathbb{Y}$ can be $\mathbb{I}\mathbb{R}$ for $\boldsymbol{f} : \mathbb{I}\mathbb{R}^d \to \mathbb{I}\mathbb{R}$
- $\mathbb{I}\mathbb{Y}$ can be $\mathbb{I}\mathbb{R}^m$ for $\boldsymbol{f} : \mathbb{I}\mathbb{R}^d \to \mathbb{I}\mathbb{R}^m$
- $\mathbb{I}\mathbb{Y}$ can be $(\mathbb{I}\mathbb{R}, \mathbb{I}\mathbb{R}^m, \mathbb{I}\mathbb{R}^{m^2})$ for range, gradient & Hessian of $\boldsymbol{f} : \mathbb{I}\mathbb{R}^d \to \mathbb{I}\mathbb{R}$
- Other obvious extensions include arithmetic over Taylor polynomial inclusion algebras
- In general the domain and range of $\boldsymbol{f}$ can be complete lattices with intervals and bisection operations
- We have seen several statistical applications of $\mathbb{Y}$-MRPs
- CODE: *mrs: a C++ class library for statistical set processing* by Bycroft, Harlow, Sainudiin, Teng and York.

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Thank you!