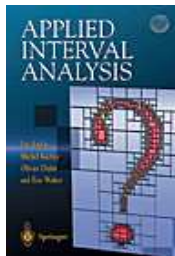


Interval analysis and robotics

Luc Jaulin, Labsticc, IHSEV, ENSTA-Bretagne

<http://www.ensta-bretagne.fr/jaulin/>

SCAN 2012, September 23, Novosibirsk



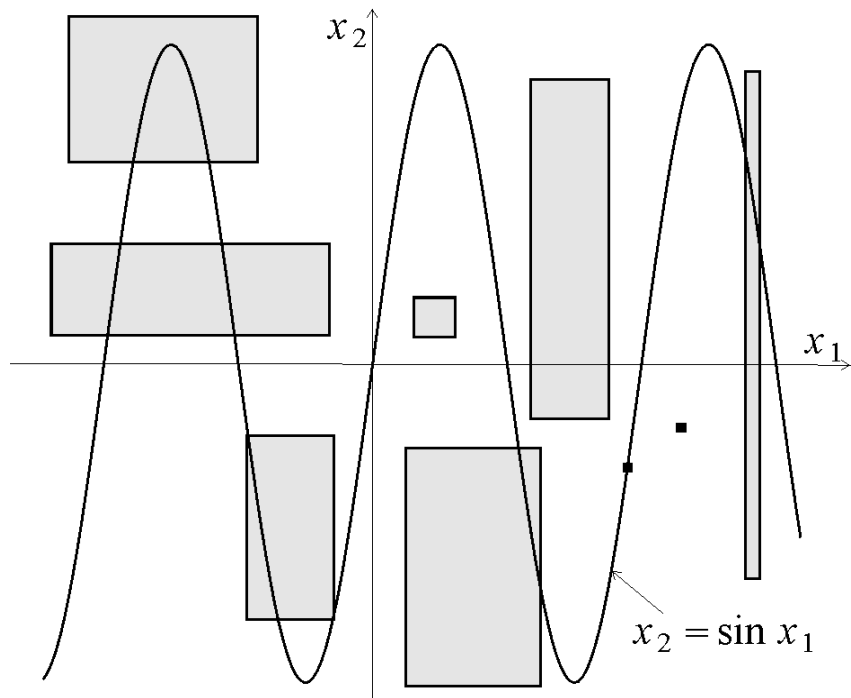
1 Contractors

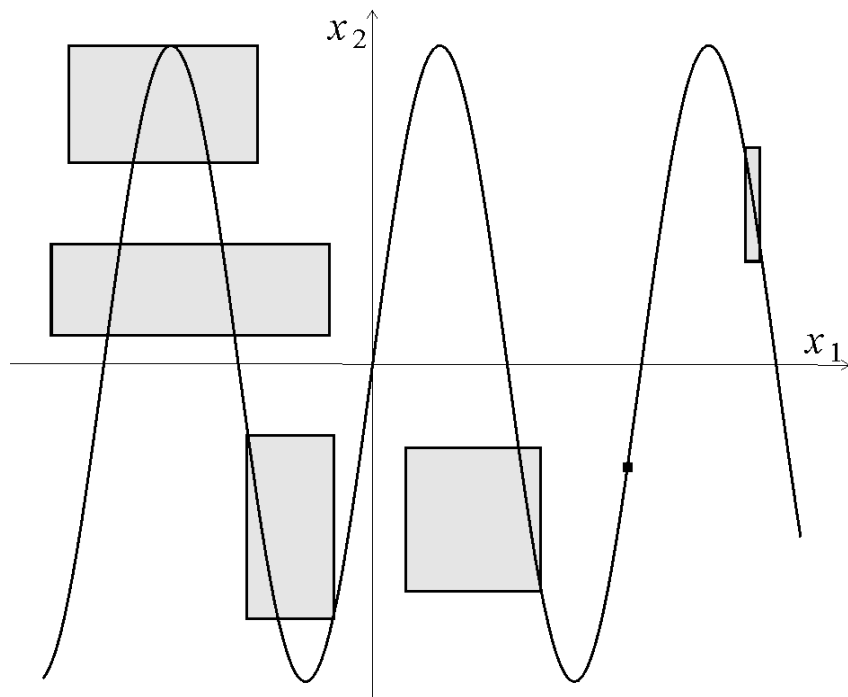
The operator $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\left\{ \begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{array} \right.$$

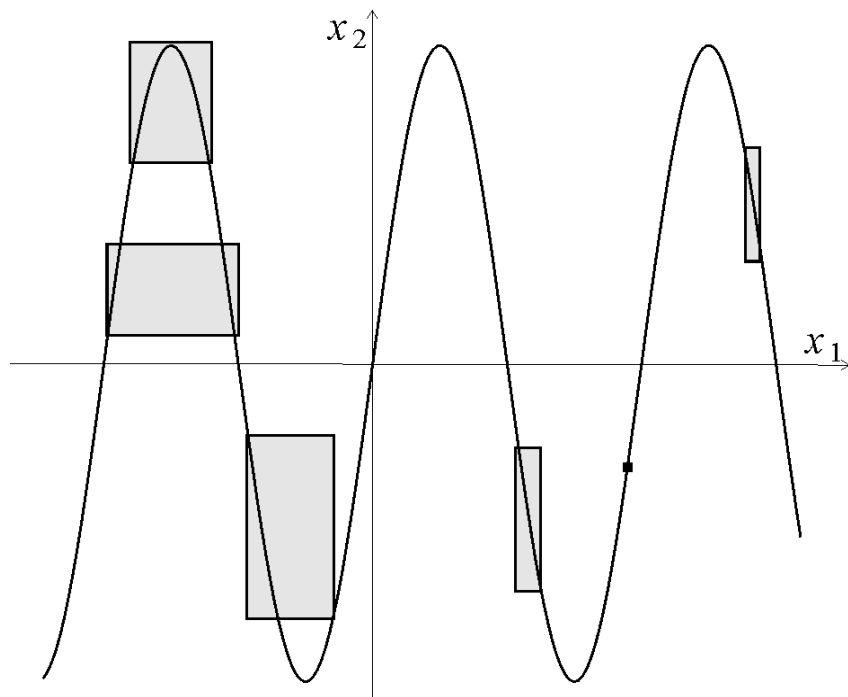
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$





Forward contraction



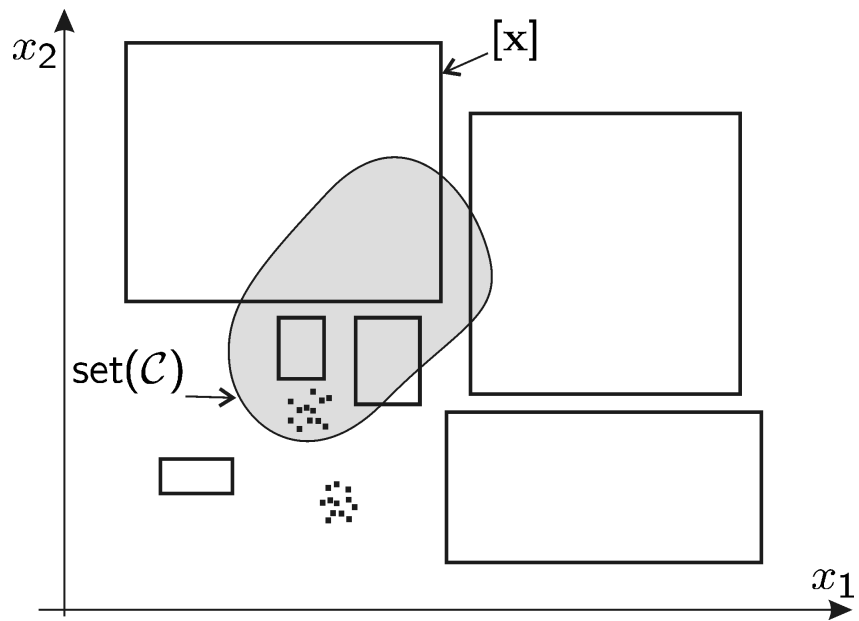
Backward contraction

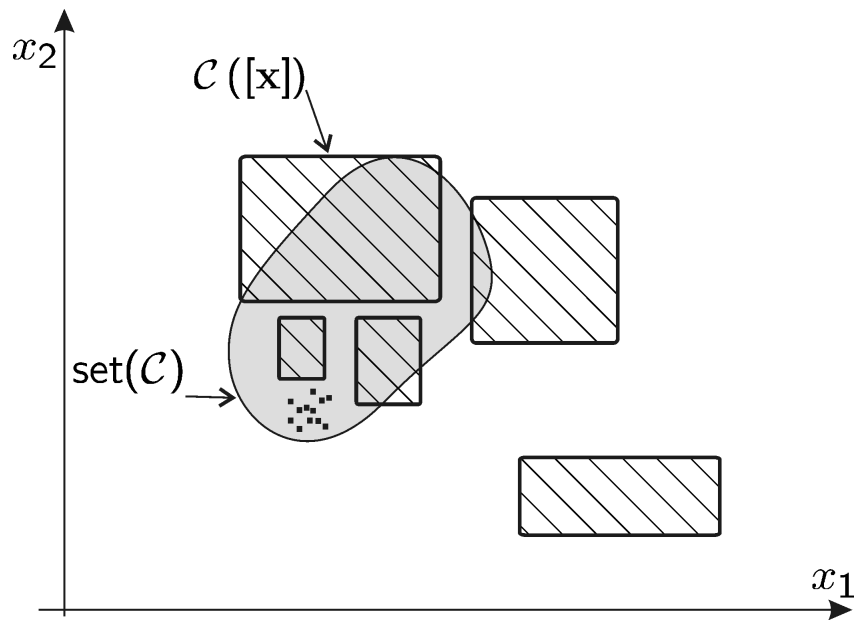
More generally, $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *contractor* if

- (i) $\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$ (contractance)
- (ii) $(\mathbf{a} \in [\mathbf{x}], \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}) \Rightarrow \mathbf{a} \in \mathcal{C}([\mathbf{x}])$ (consistence)

The set associated to $\mathcal{C}$ is

$$\text{set}(\mathcal{C}) = \{\mathbf{a} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}\}.$$





$\mathcal{C}$ is <i>monotonic</i> if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is <i>minimal</i> if	$\mathcal{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \text{set}(\mathcal{C})]$
$\mathcal{C}$ is <i>idempotent</i> if	$\mathcal{C}(\mathcal{C}([\mathbf{x}])) = \mathcal{C}([\mathbf{x}])$
$\mathcal{C}$ is <i>continuous</i> if	$\mathcal{C}(\mathcal{C}^\infty([\mathbf{x}])) = \mathcal{C}^\infty([\mathbf{x}])$.

Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1 ([\mathbf{x}]) \cap \mathcal{C}_2 ([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1 ([\mathbf{x}]) \cup \mathcal{C}_2 ([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1 (\mathcal{C}_2 ([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Dealing with outliers

$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Properties

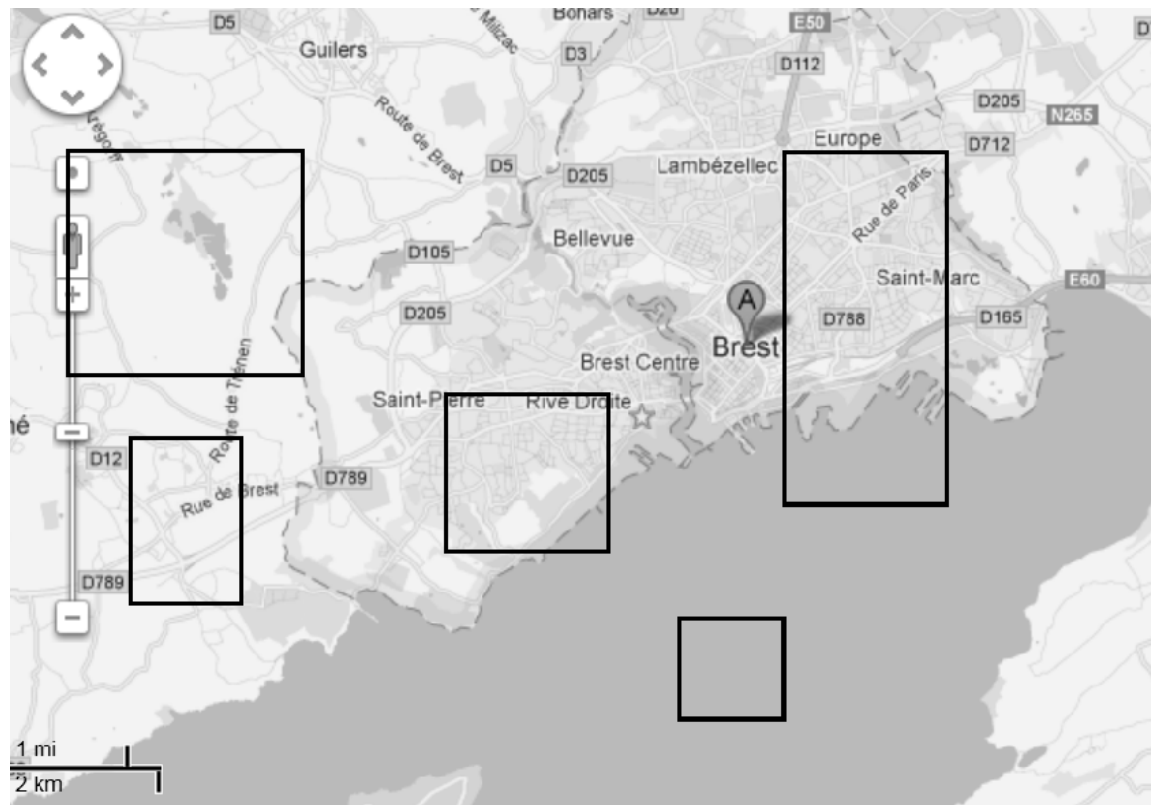
$$(\mathcal{C}_1^\infty \cap \mathcal{C}_2^\infty)^\infty = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$$

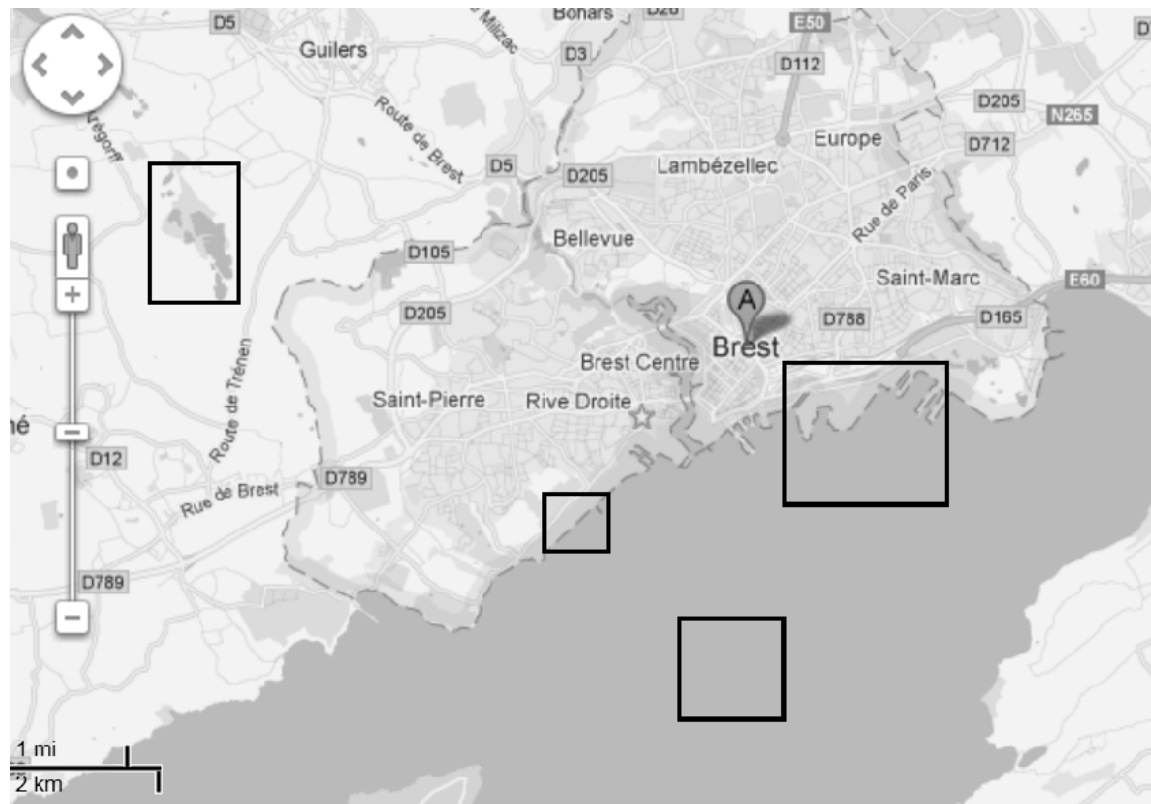
$$(\mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)) \supset (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

$$\begin{cases} \mathcal{C}_1 \text{ minimal} \\ \mathcal{C}_2 \text{ minimal} \end{cases} \Rightarrow \mathcal{C}_1 \cup \mathcal{C}_2 \text{ minimal}$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.





Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$\begin{aligned}x_3 = x_1 + x_2 &\Rightarrow x_3 \in [x_3] \cap ([x_1] + [x_2]) && // \text{ forward} \\x_1 = x_3 - x_2 &\Rightarrow x_1 \in [x_1] \cap ([x_3] - [x_2]) && // \text{ backward} \\x_2 = x_3 - x_1 &\Rightarrow x_2 \in [x_2] \cap ([x_3] - [x_1]) && // \text{ backward}\end{aligned}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C} \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

Forward-backward contractor (HC4 revise)

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2],$$

we have the following contractor:

algorithm $\mathcal{C}$ (inout $[x_1], [x_2], [x_3]$)	
$[a] = [x_1] + [x_2]$	// $a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	// $b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	// $b \in [1, 2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	// $x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	// $a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	// $x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	// $x_2 = a - x_1$

2 Solver

Example 1. Solve the system

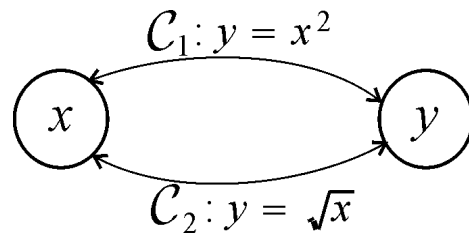
$$y = x^2$$

$$y = \sqrt{x}.$$

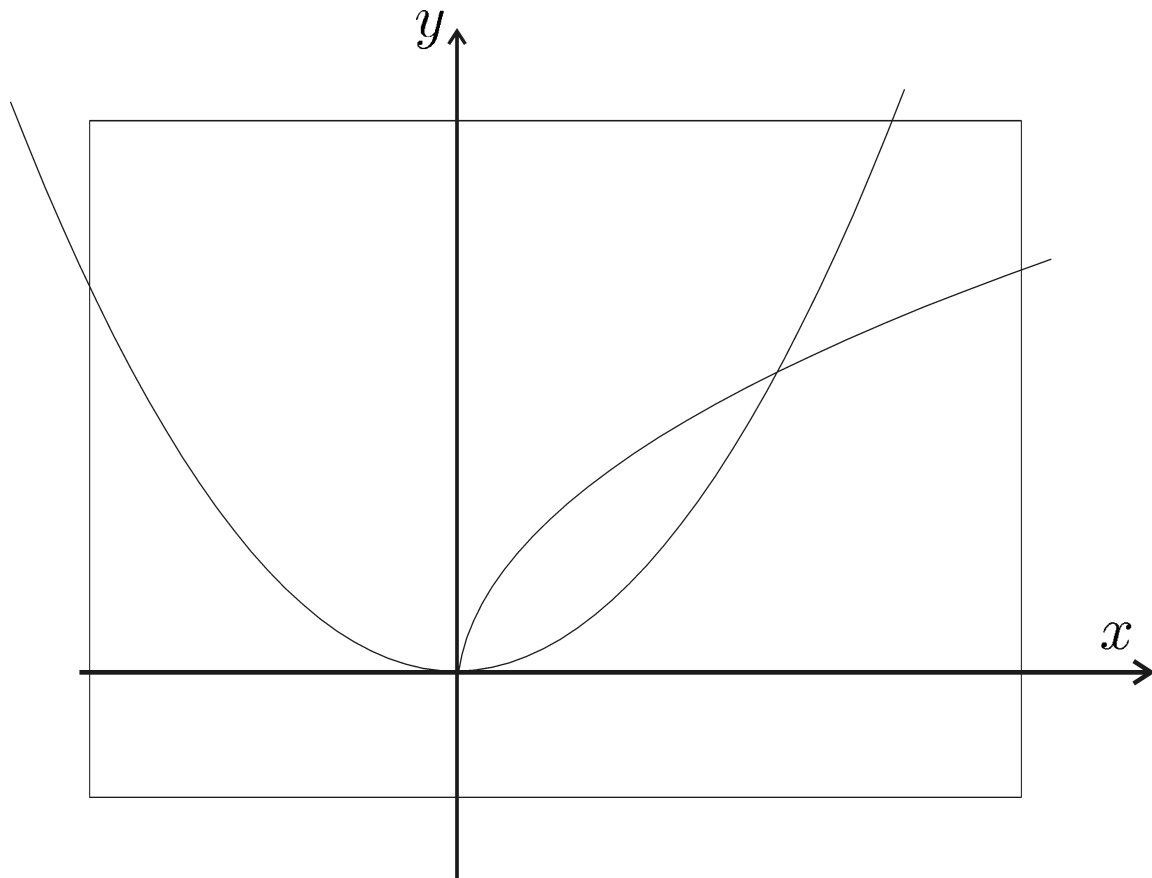
We build two contractors

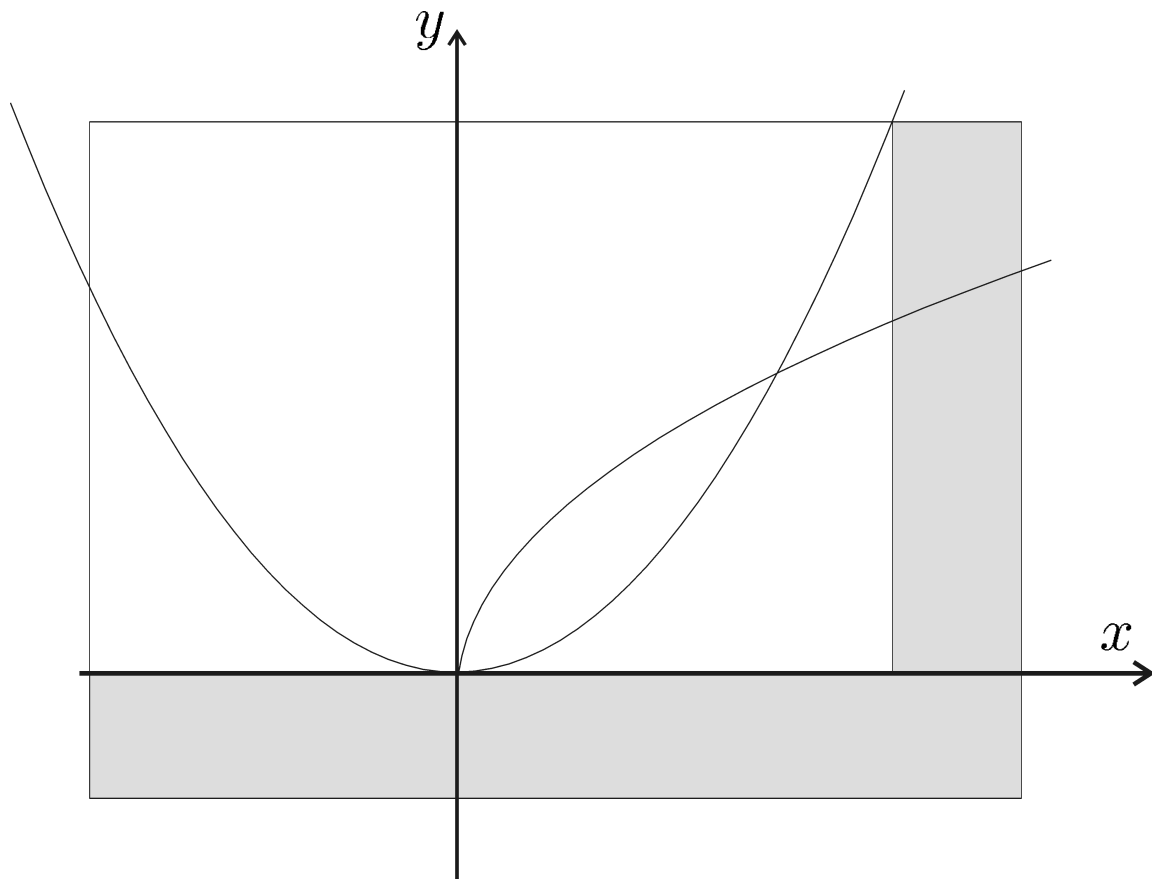
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

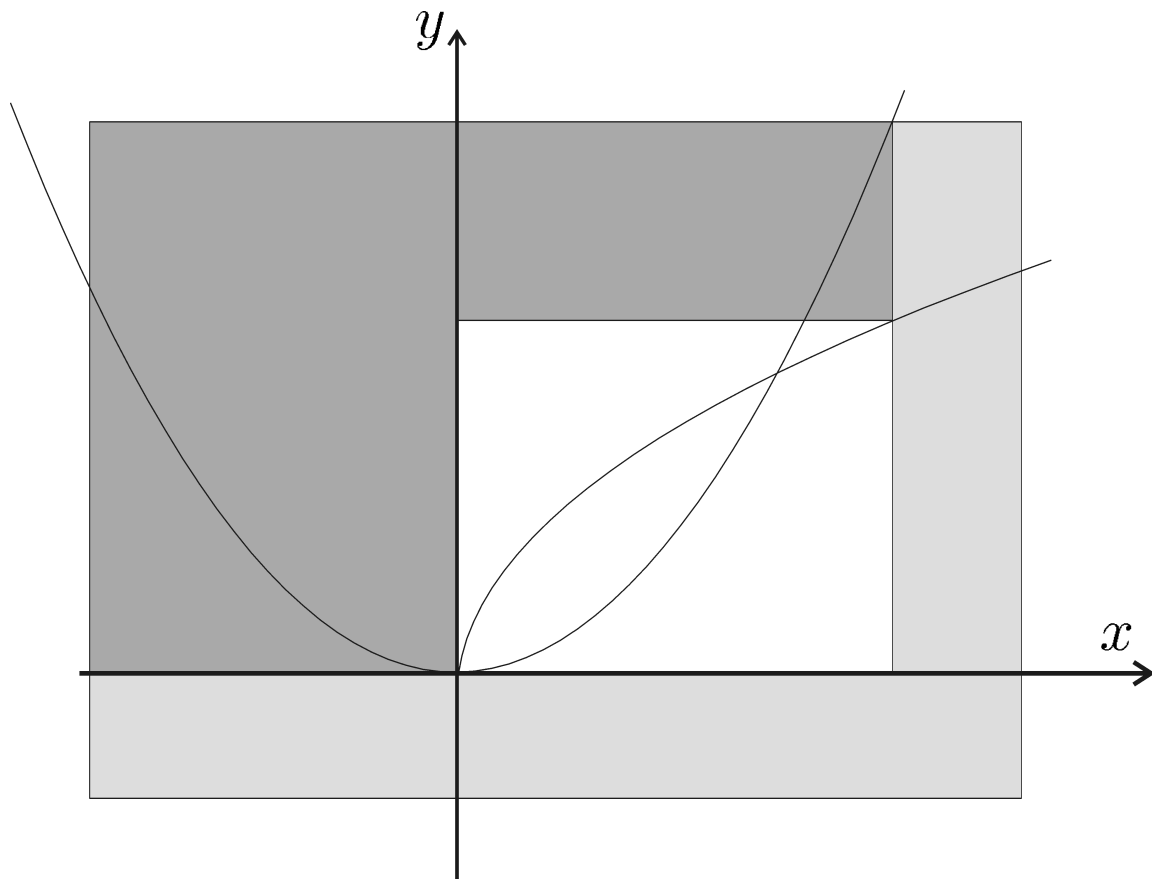
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$

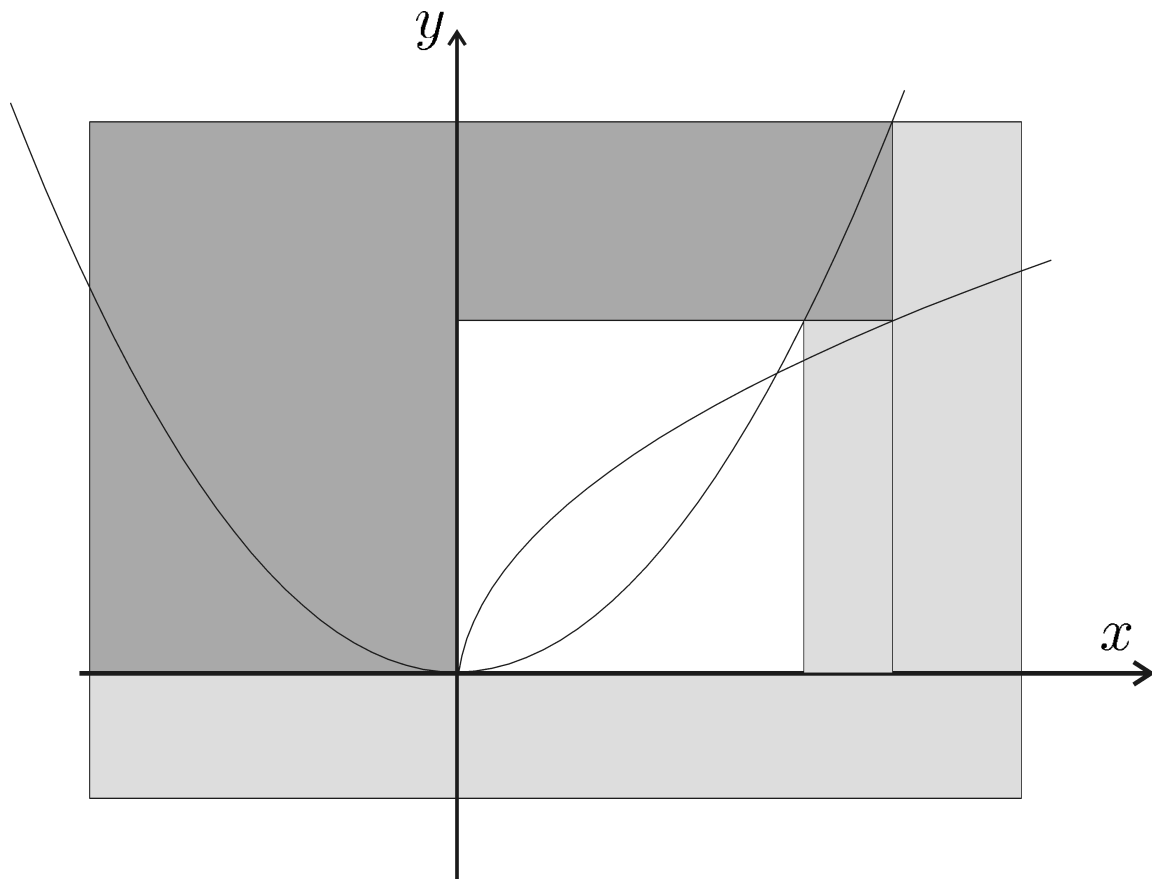


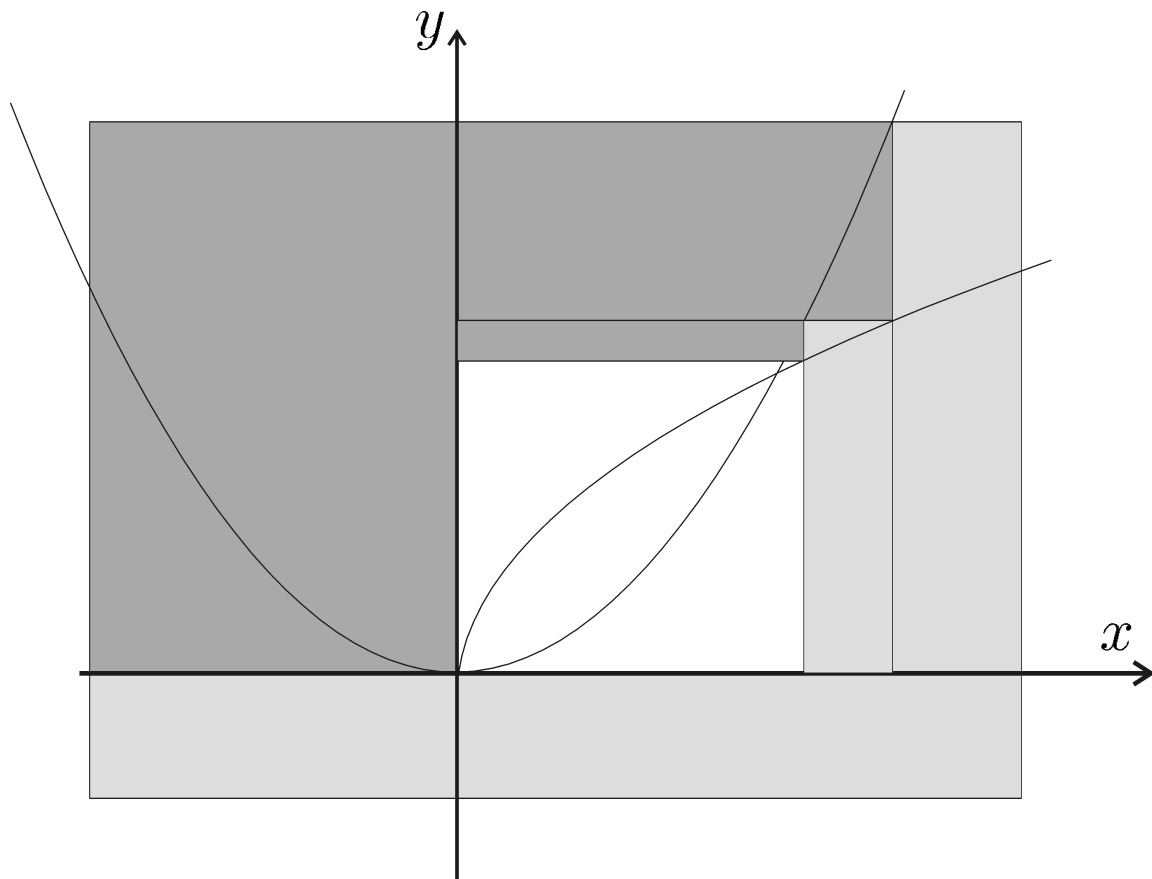
Contractor graph

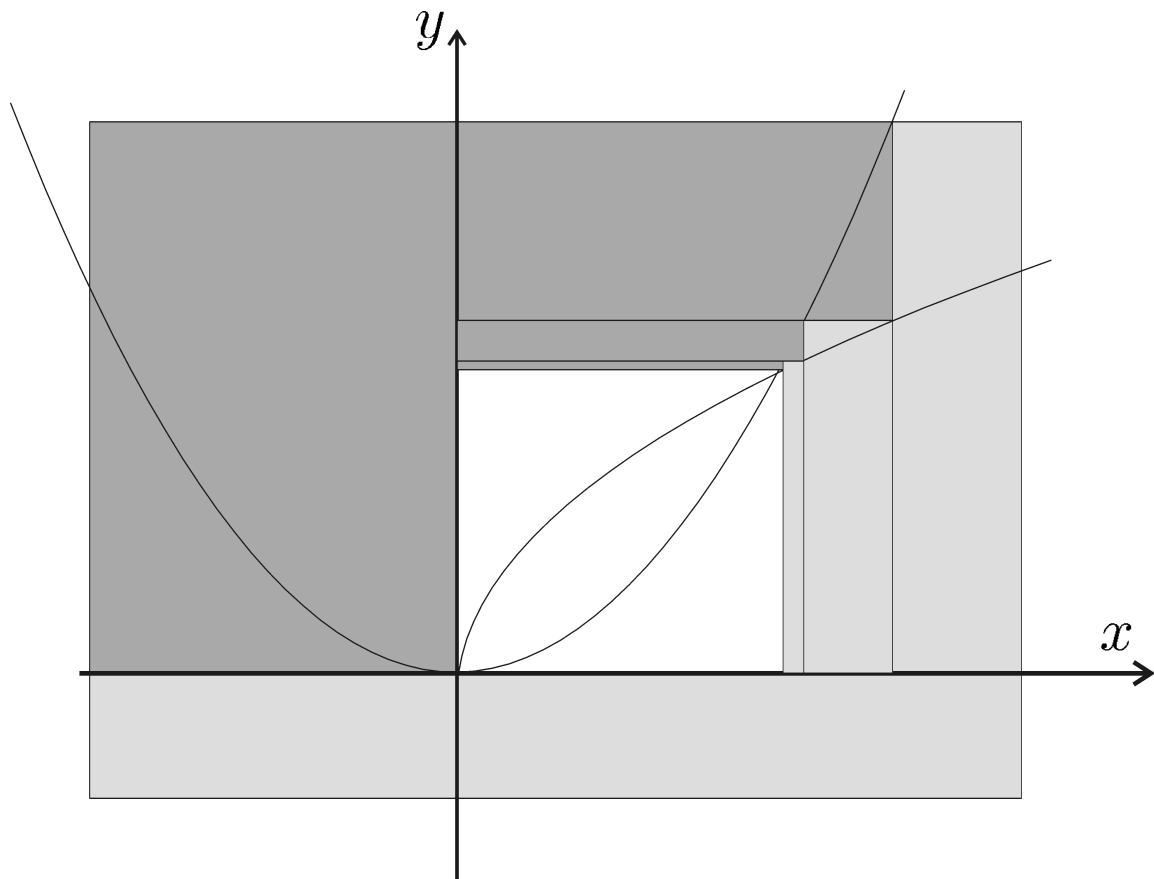


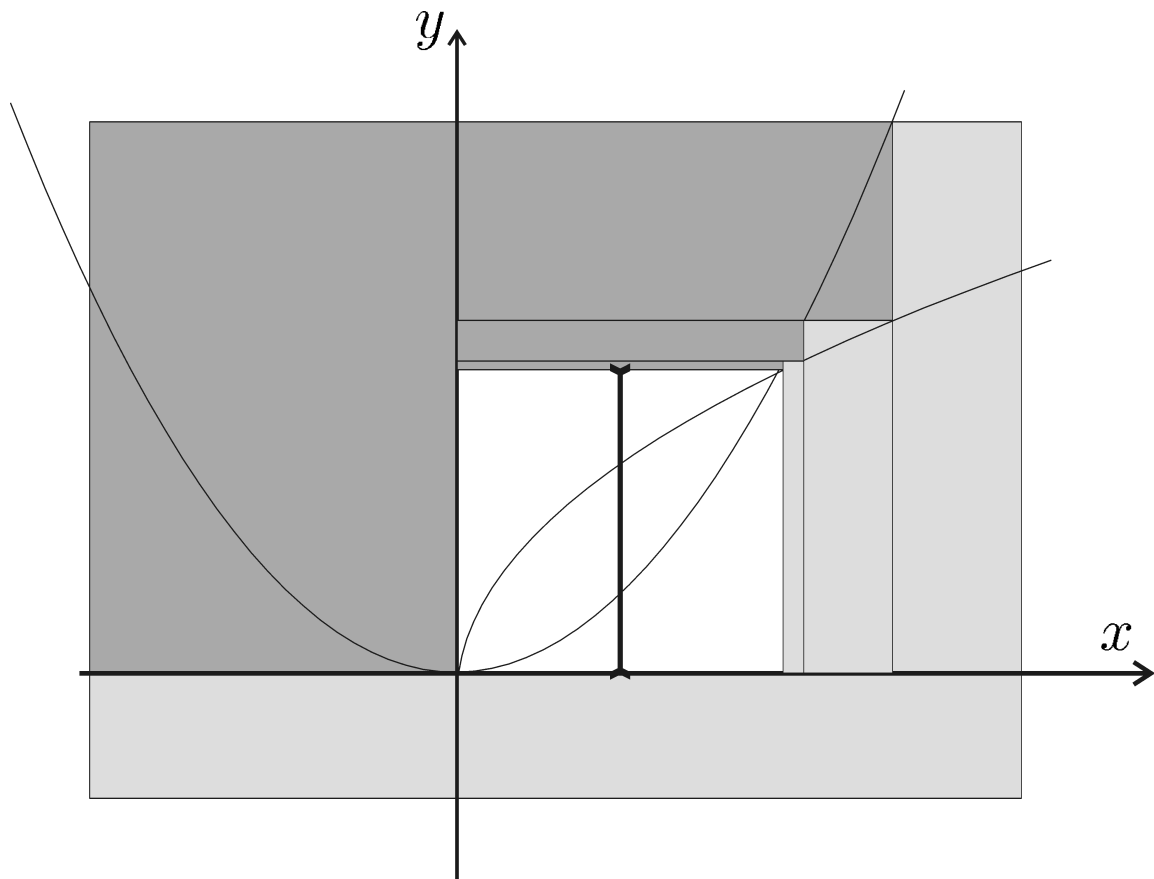


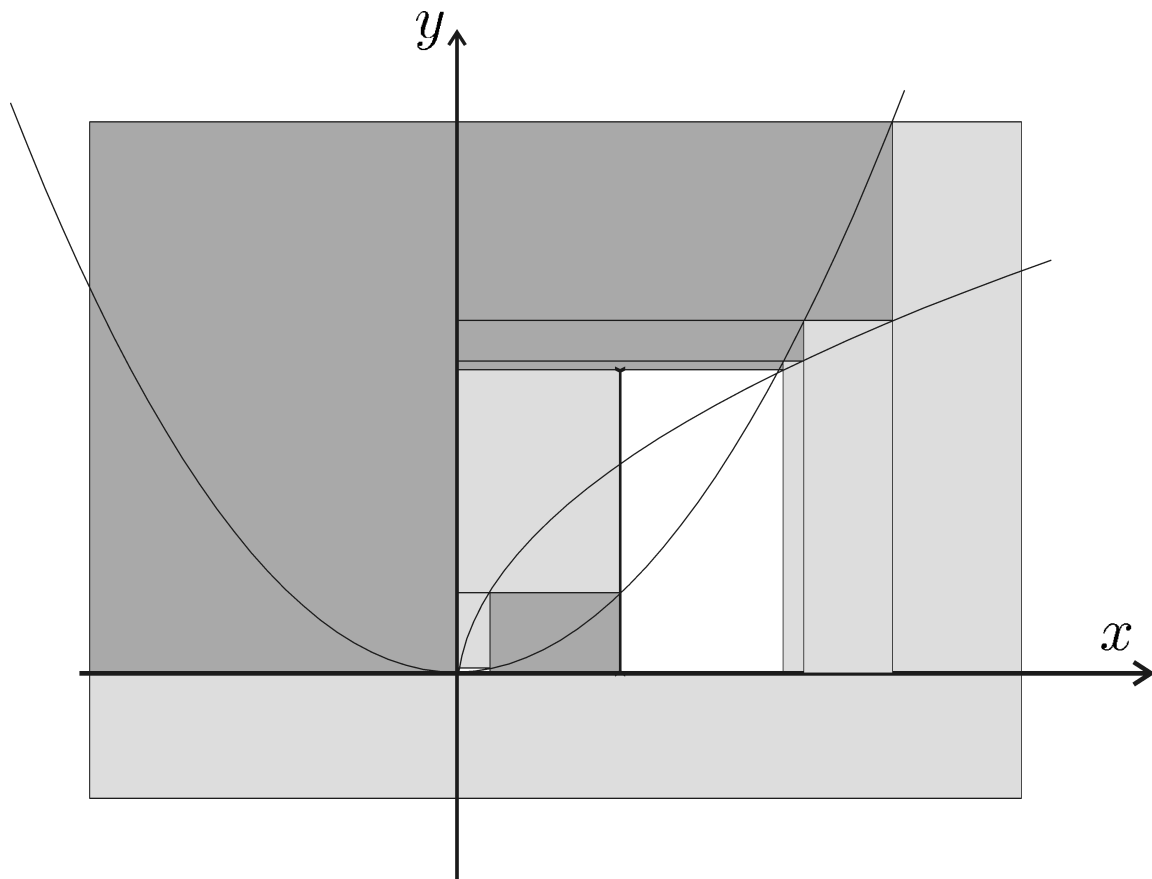


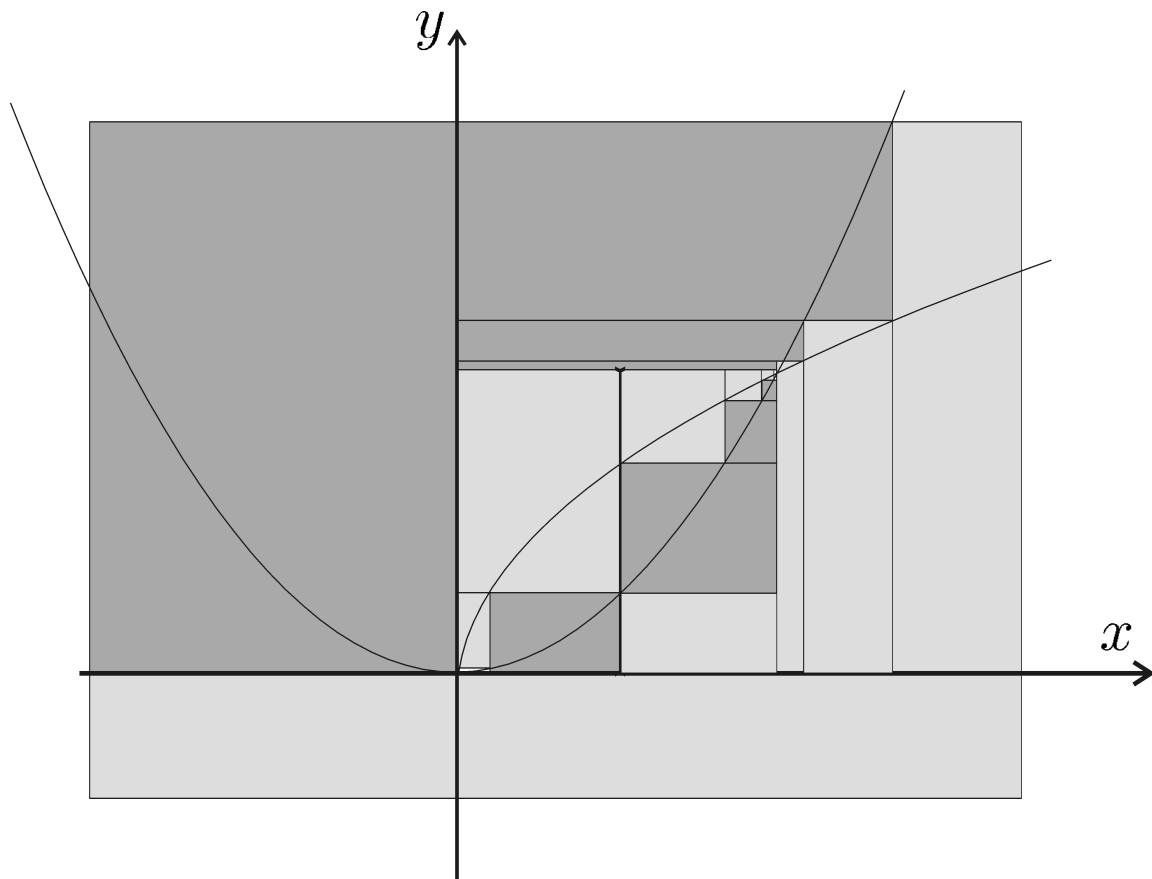












Note that

$\mathcal{C}_1$ is optimal

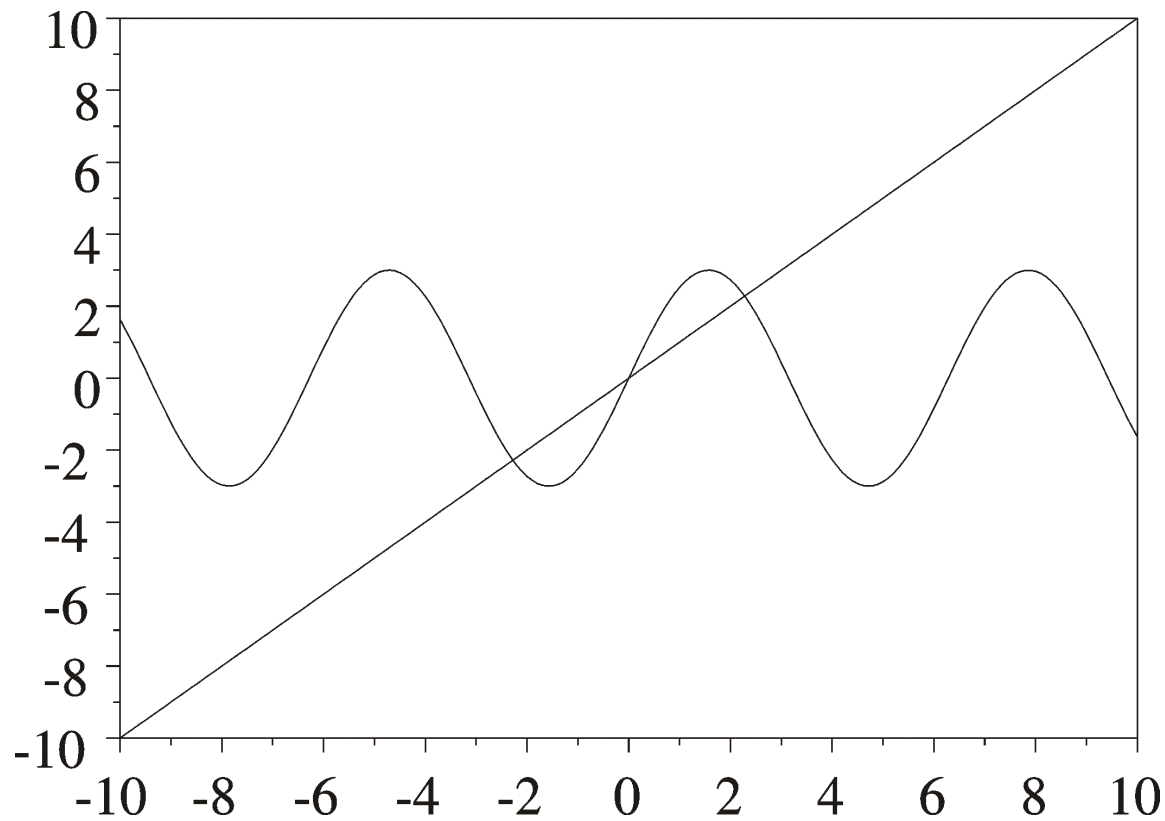
$\mathcal{C}_2$ is optimal

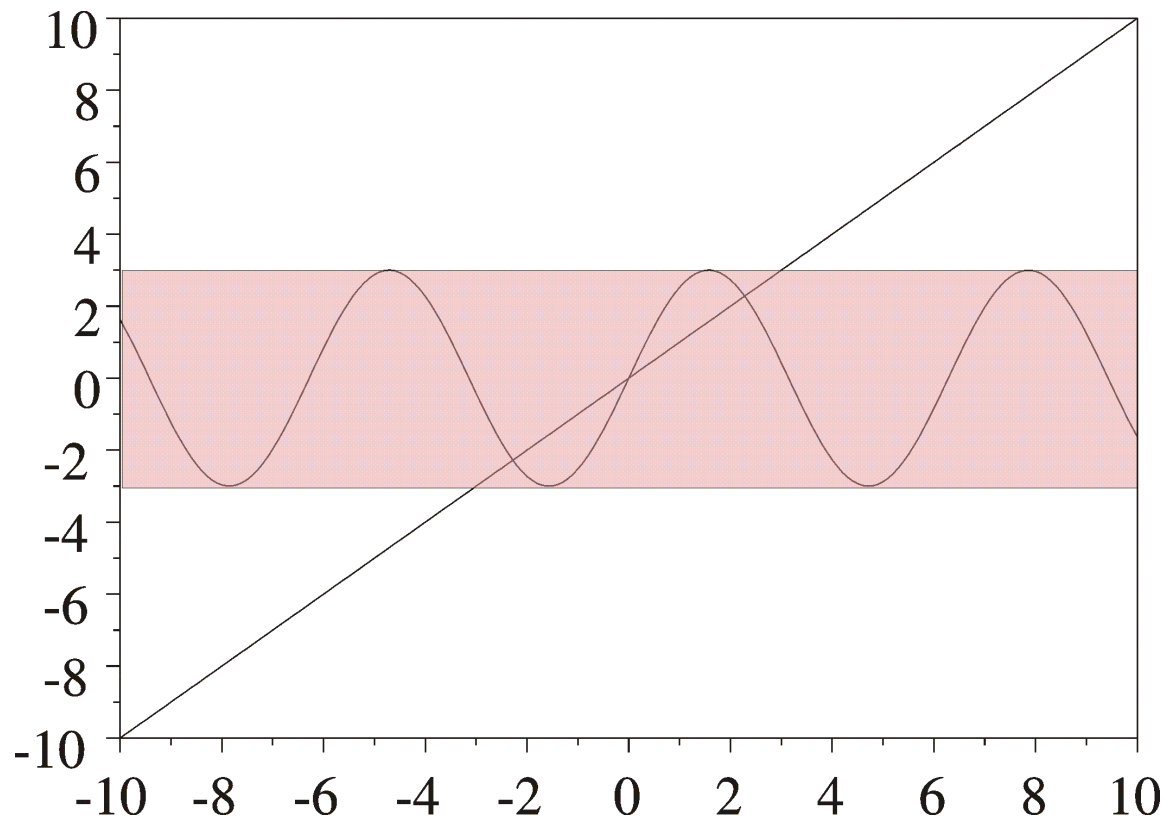
$\mathcal{C}_1 \circ \mathcal{C}_2$ is not optimal

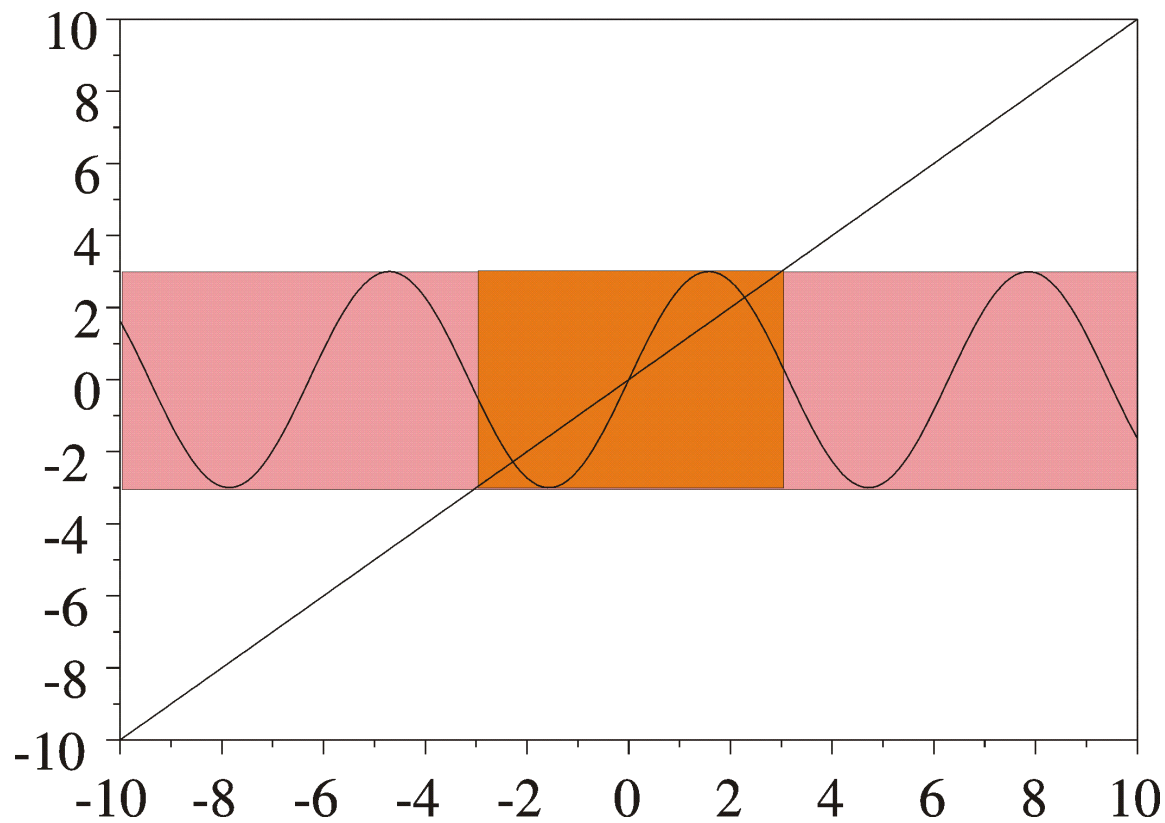
$(\mathcal{C}_1 \circ \mathcal{C}_2)^\infty$ is optimal.

Example 2. Consider the system

$$\begin{cases} y = 3 \sin(x) \\ y = x \end{cases} \quad x \in \mathbb{R}, y \in \mathbb{R}.$$

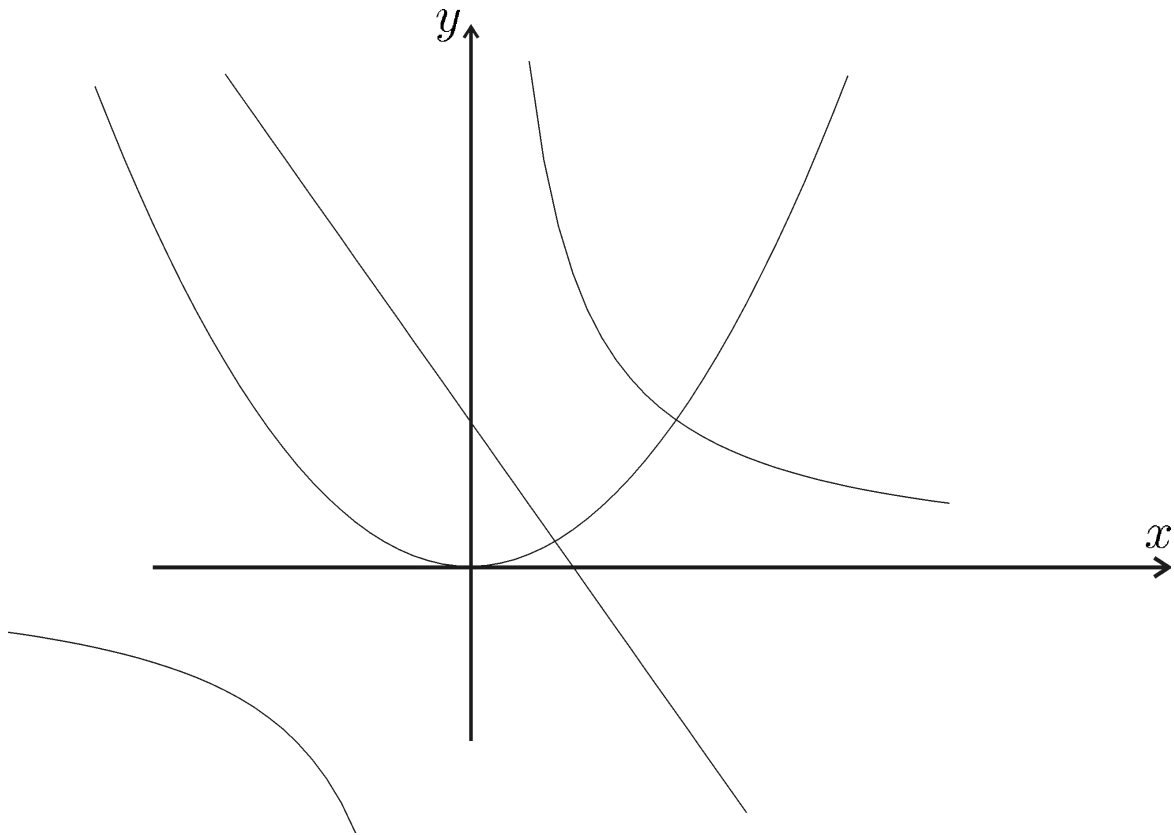


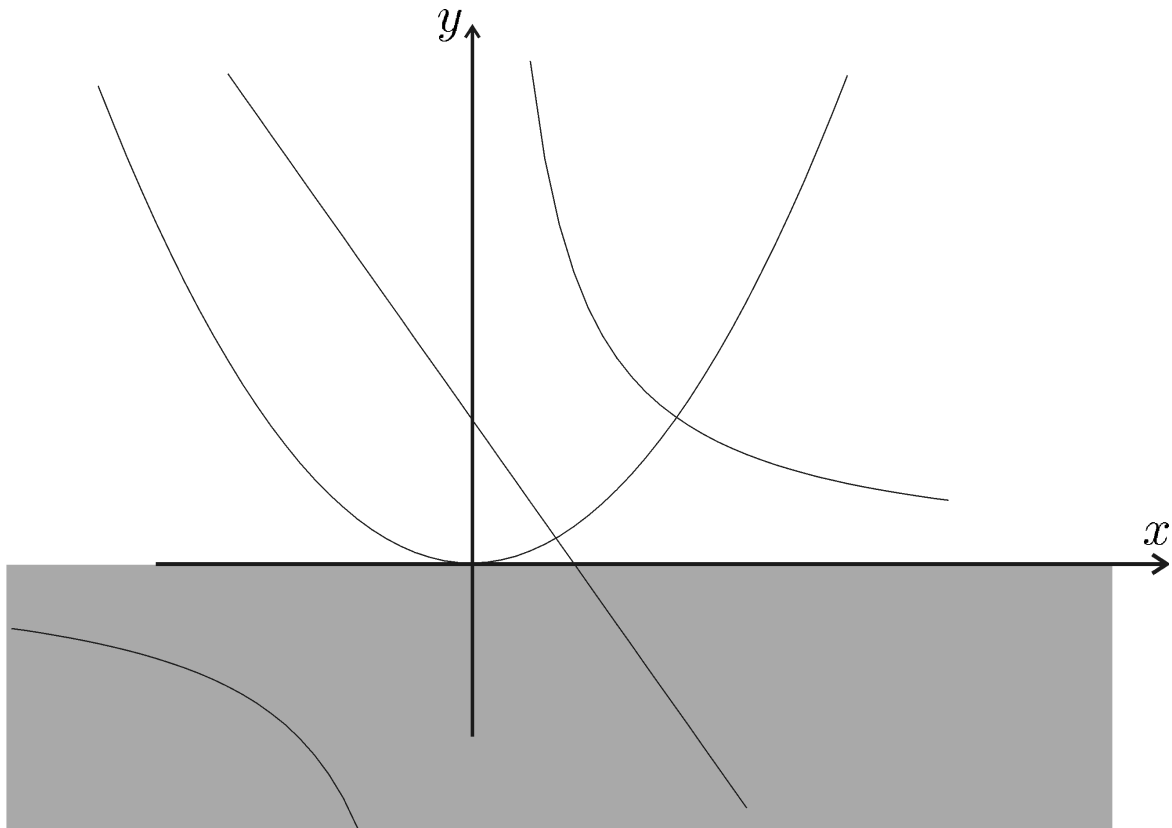


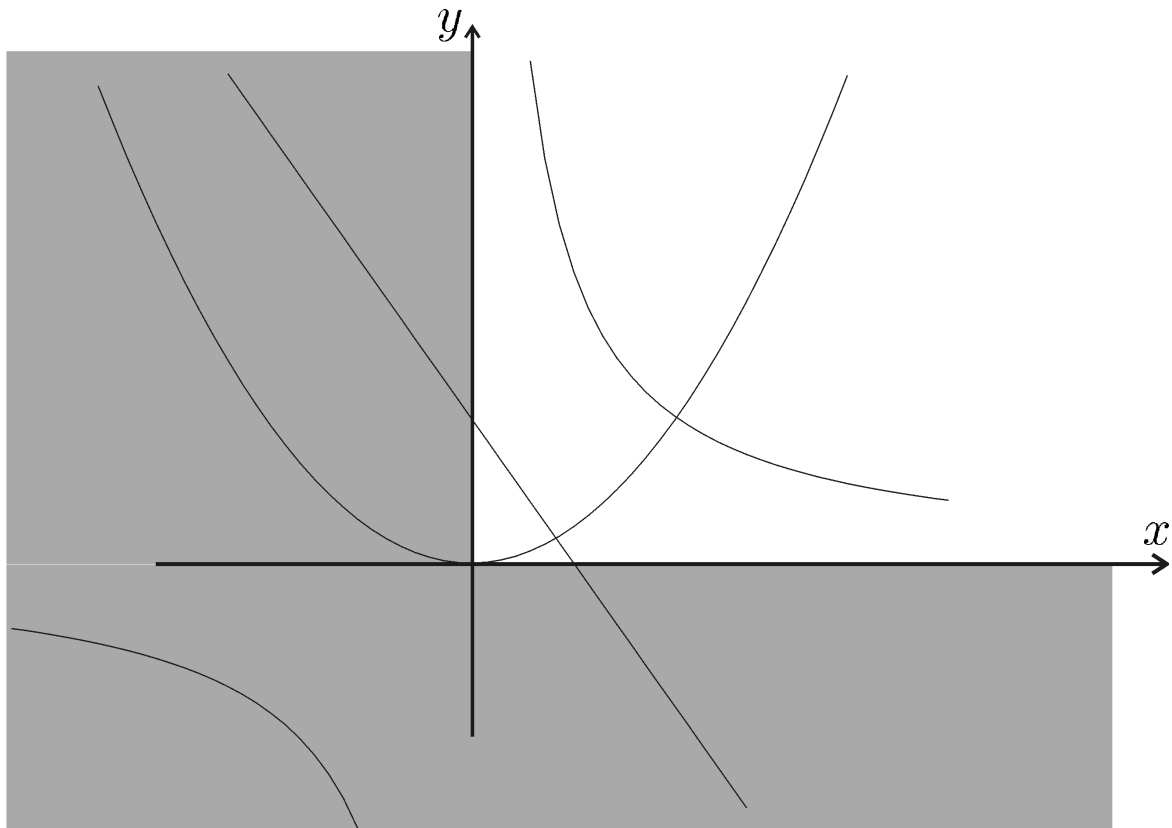


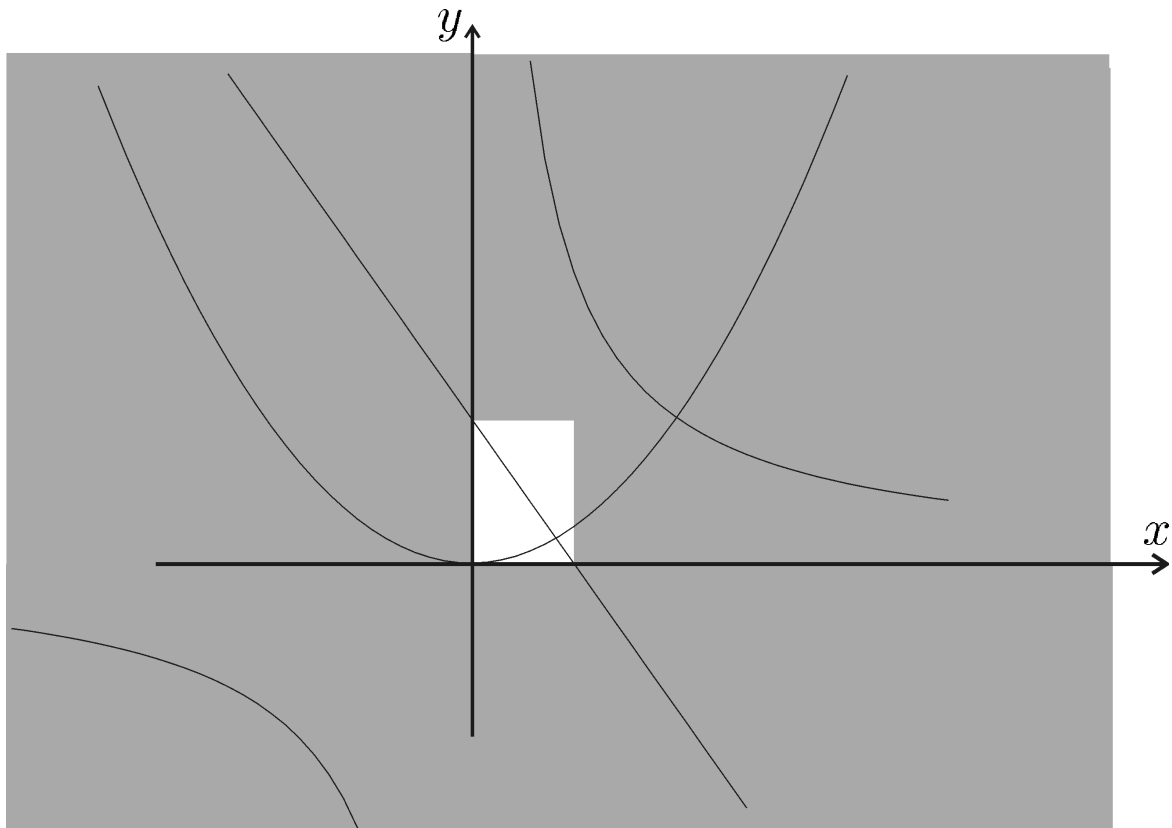
Example 3. Consider the following problem

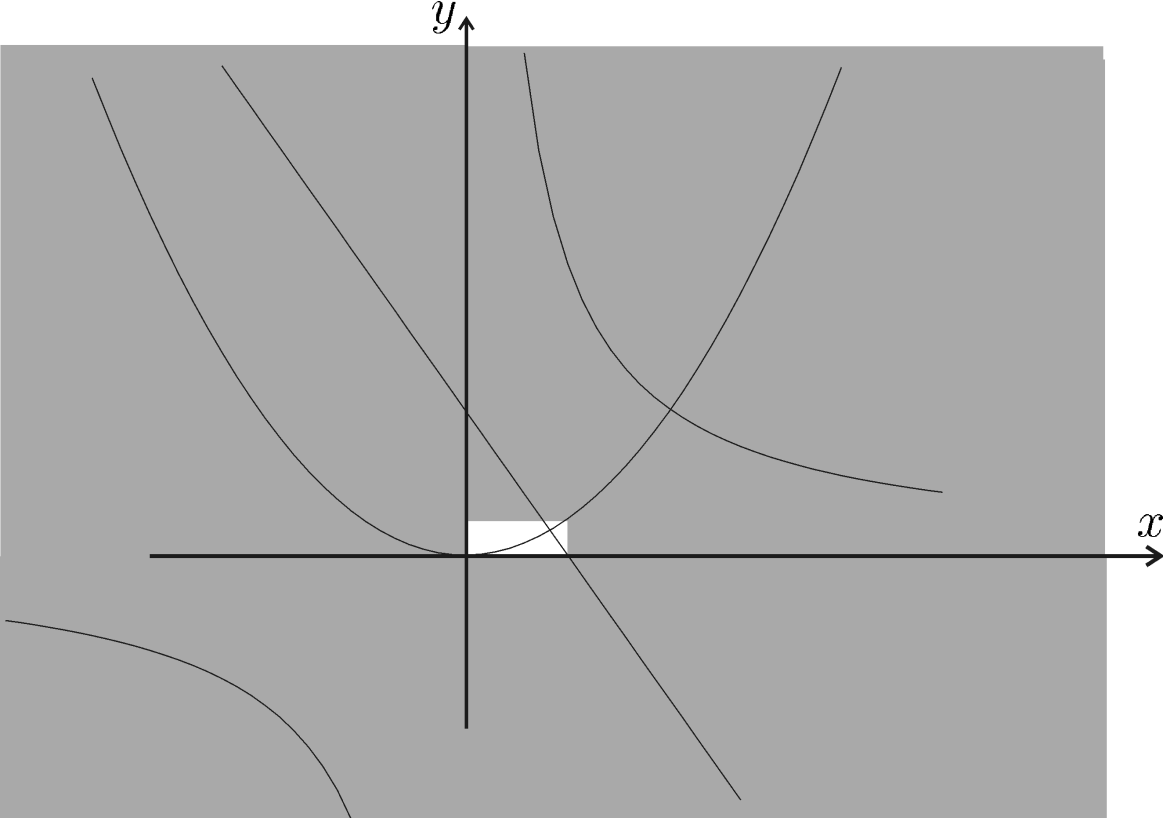
$$\begin{cases} (C_1) : & y = x^2 \\ (C_2) : & xy = 1 \\ (C_3) : & y = -2x + 1 \end{cases}$$

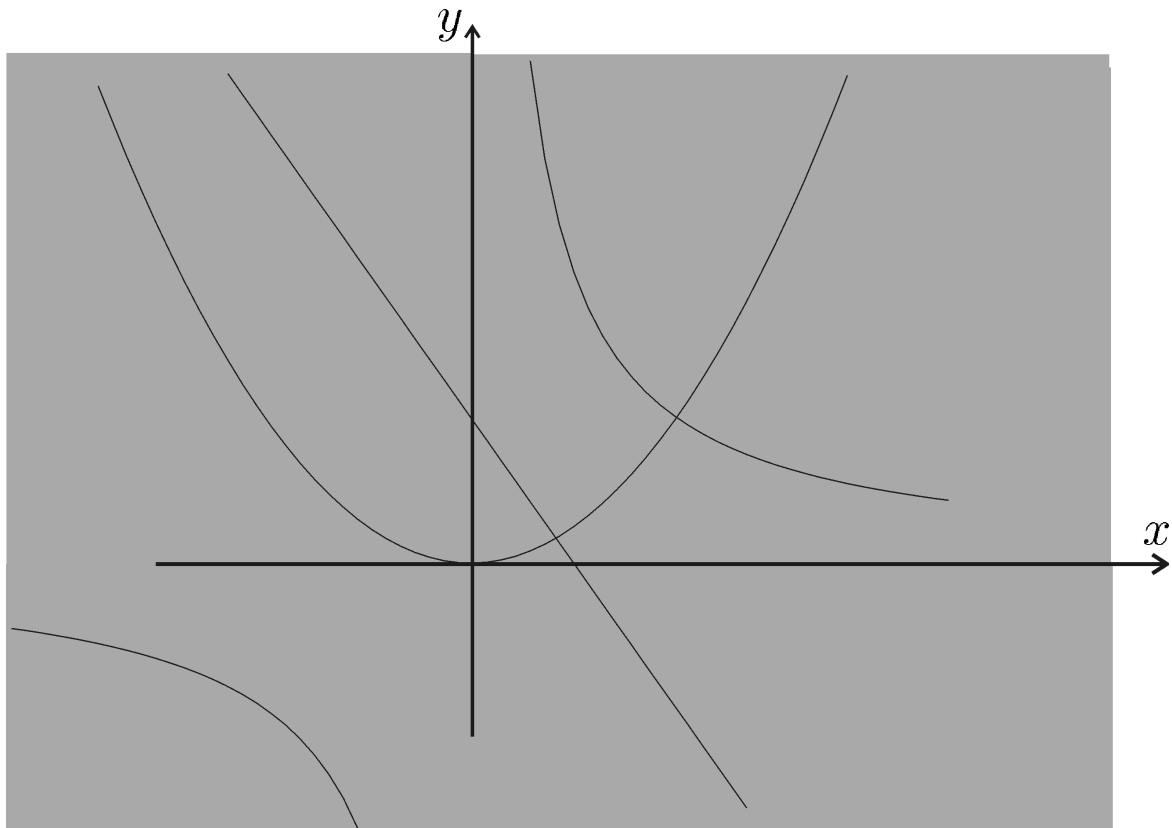












$$(C_1) \Rightarrow y \in [-\infty, \infty]^2 = [0, \infty]$$

$$(C_2) \Rightarrow x \in 1/[0, \infty] = [0, \infty]$$

$$(C_3) \Rightarrow y \in [0, \infty] \cap ((-2) \cdot [0, \infty] + 1) \\ = [0, \infty] \cap (-\infty, 1] = [0, 1]$$

$$x \in [0, \infty] \cap (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}]$$

$$(C_1) \Rightarrow y \in [0, 1] \cap [0, 1/2]^2 = [0, 1/4]$$

$$(C_2) \Rightarrow x \in [0, 1/2] \cap 1/[0, 1/4] = \emptyset$$

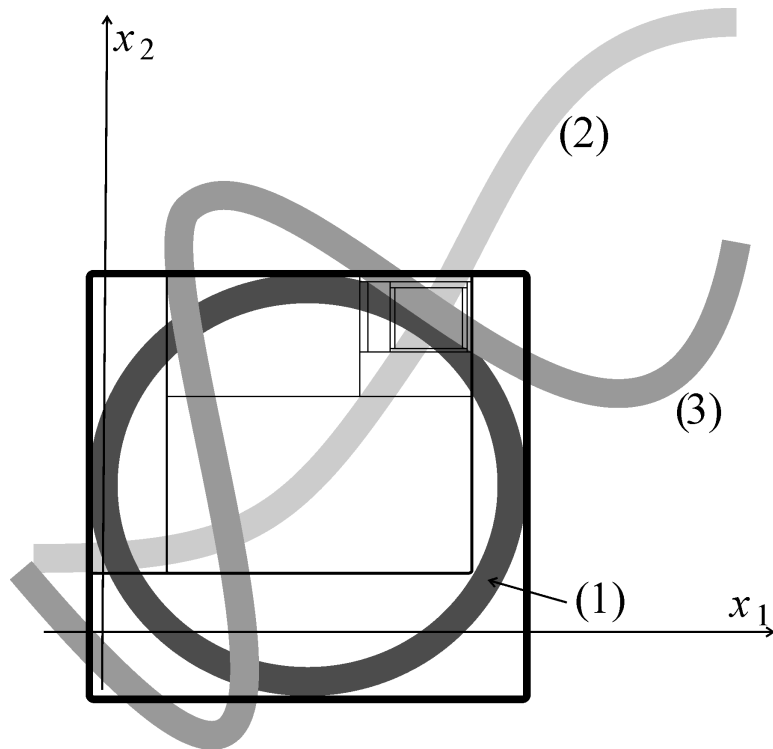
$$y \in [0, 1/4] \cap 1/\emptyset = \emptyset$$

3 Redundant system of equations

Problem

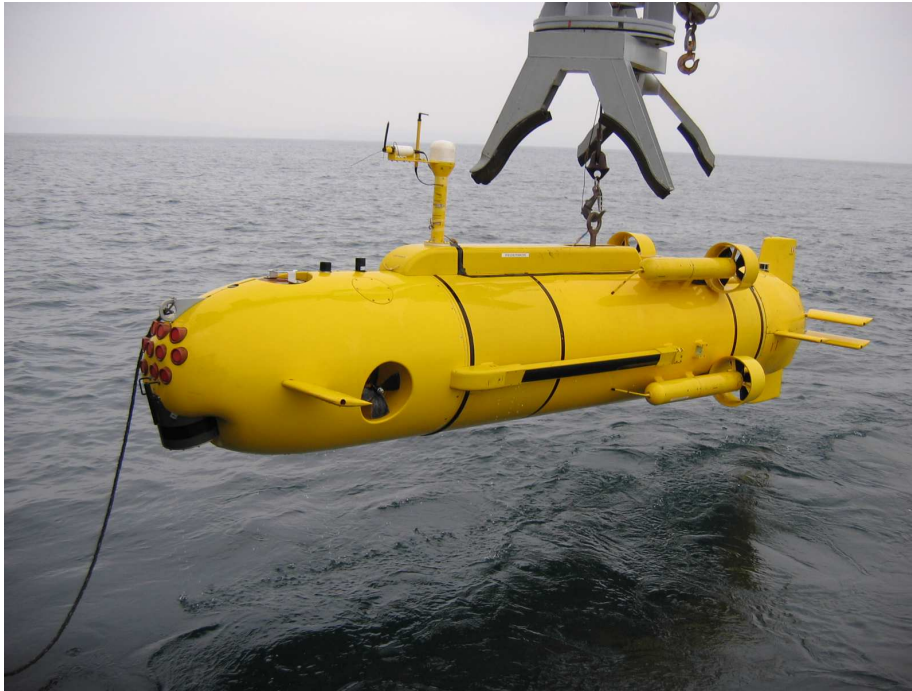
$$\left\{ \begin{array}{l} f_i(\mathbf{x}, \mathbf{y}_i) = 0, \\ \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y}_i \in [\mathbf{y}_i] \subset \mathbb{R}^{p_i} \quad . \\ i \in \{1, \dots, m\} \end{array} \right.$$

with $m \gg n \gg 1$.

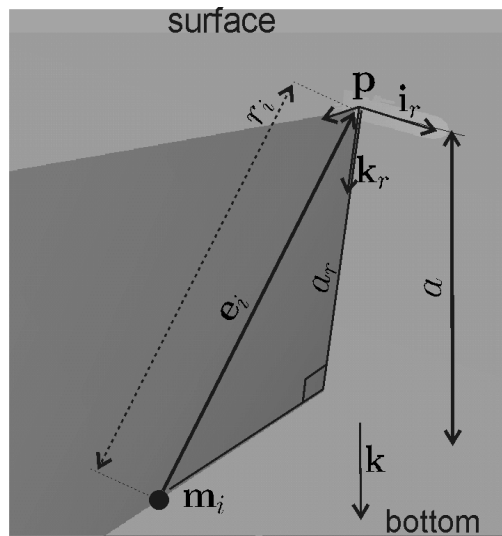
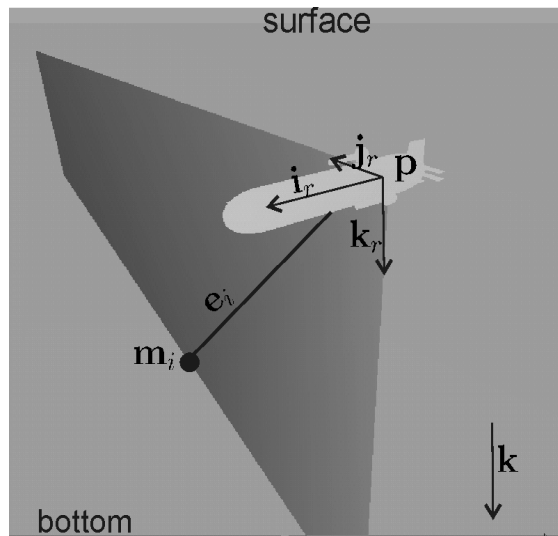


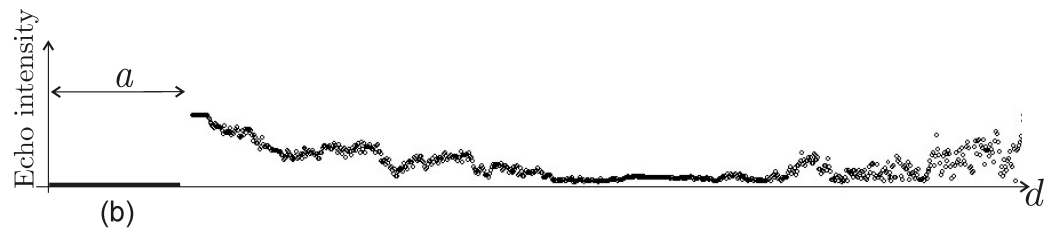
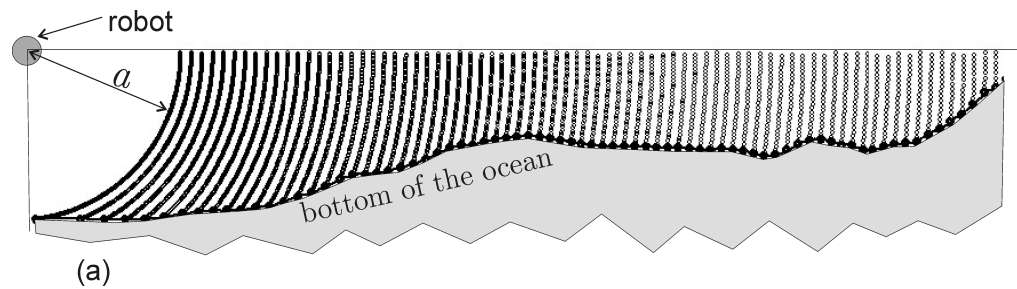
4 SLAM

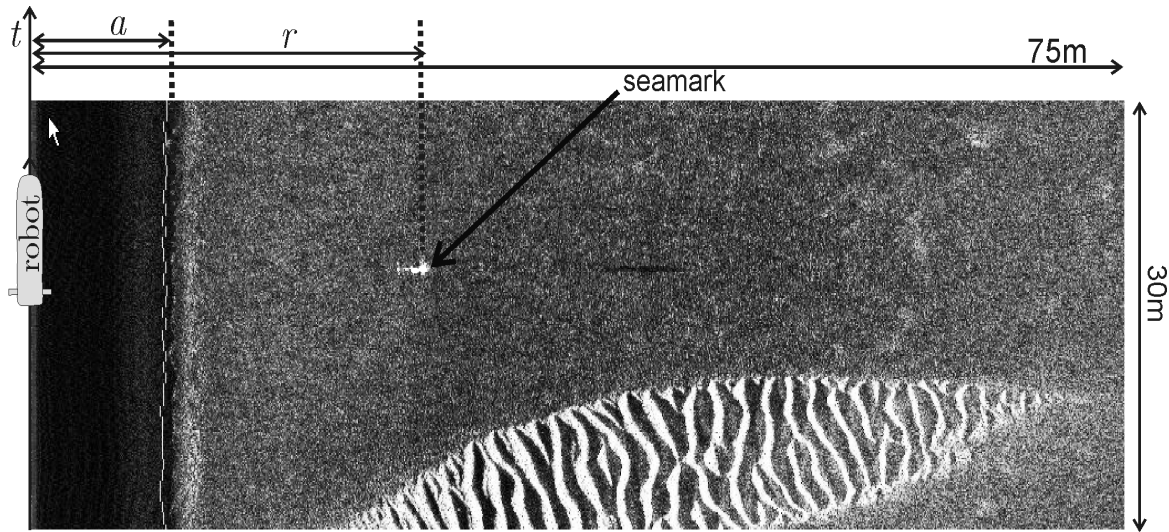
L. Jaulin (2009), A nonlinear set-membership approach for the localization and map building of an underwater robot using interval constraint propagation, IEEE Transactions on Robotics.



Show the video







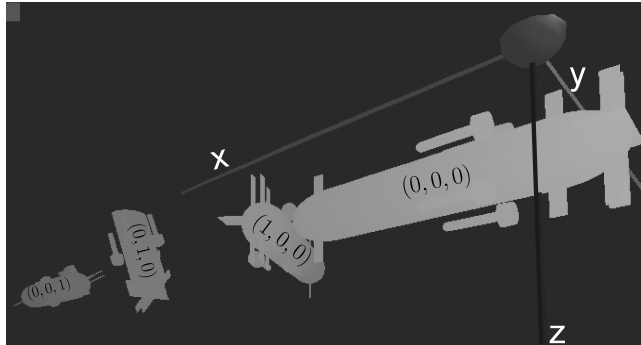
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1, 1] . \tilde{\mathbf{v}}_r + 0.004 * [-1, 1]$$

Inertial central (Octans III from IXSEA).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4} \cdot [-1, 1] \\ 1.75 \times 10^{-4} \cdot [-1, 1] \\ 5.27 \times 10^{-3} \cdot [-1, 1] \end{pmatrix}.$$



Six mines have been detected.

i	0	1	2	3	4	5
$\tau(i)$	7054	7092	7374	7748	9038	9688
$\sigma(i)$	1	2	1	0	1	5
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98

6	7	8	9	10	11
10024	10817	11172	11232	11279	11688
4	3	3	4	5	1
37.90	36.71	37.37	31.03	33.51	15.05

4.1 Constraints

$$t \in \{6000.0, 6000.1, 6000.2, \dots, 11999.4\},$$

$$i \in \{0, 1, \dots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \cdot \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) \cdot \frac{\pi}{180}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos \psi(t) & -\sin \psi(t) & 0 \\ \sin \psi(t) & \cos \psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos \theta(t) & 0 & \sin \theta(t) \\ 0 & 1 & 0 \\ -\sin \theta(t) & 0 & \cos \theta(t) \end{pmatrix},$$

$$\mathbf{R}_\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi(t) & -\sin \varphi(t) \\ 0 & \sin \varphi(t) & \cos \varphi(t) \end{pmatrix},$$

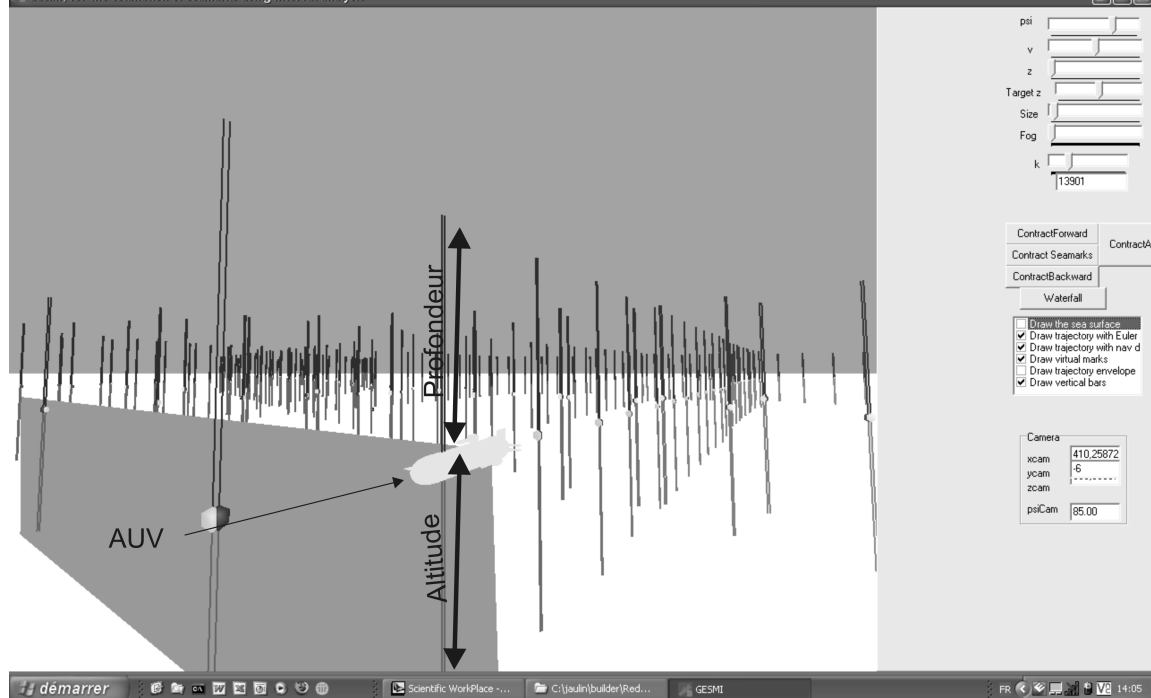
$$\mathbf{R}(t) = \mathbf{R}_\psi(t) \cdot \mathbf{R}_\theta(t) \cdot \mathbf{R}_\varphi(t),$$

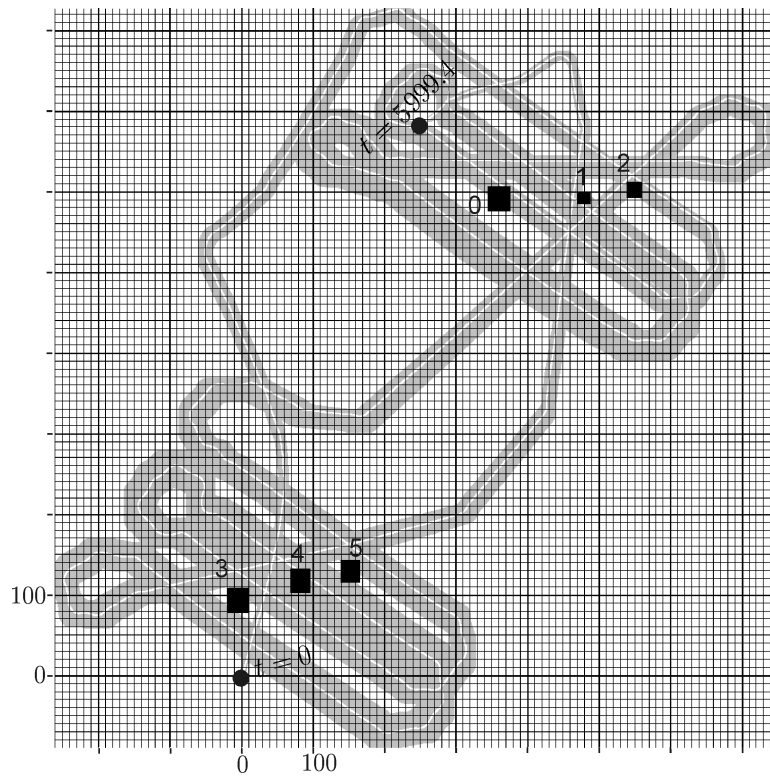
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t) \cdot \mathbf{v}_r(t),$$

$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| = r(i),$$

$$\mathbf{R}^\top(\tau(i)) \cdot (\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2}.$$

4.2 GESMI





4.3 Quimper language

```
//-----  
Constants  
N = 59996; // Number of time steps
```

```
Variables
```

```
    R[N-1][3][3], // rotation matrices  
    p[N][3], // positions  
    v[N-1][3], // speed vectors  
    phi[N-1],theta[N-1],psi[N-1]; // Euler angles  
    px[N],py[N]; // for display only
```

```
//-----
```

```
function R[3][3]=euler(phi,theta,psi)
    cphi = cos(phi);
    sphi = sin(phi);
    ctheta = cos(theta);
    stheta = sin(theta);
    cpsi = cos(psi);
    spsi = sin(psi);
    R[1][1]=ctheta*cpsi;
    R[1][2]=-cphi*spsi+stheta*cpsi*sphi;
    R[1][3]=spsi*sphi+stheta*cpsi*cphi;
    R[2][1]=ctheta*spsi;
    R[2][2]=cpsi*cphi+stheta*spsi*sphi;
    R[2][3]=-cpsi*sphi+stheta*cphi*spsi;
    R[3][1]=-stheta;
    R[3][2]=ctheta*sphi;
    R[3][3]=ctheta*cphi;
end
```

```
contractor-list rotation
```

```
    for k=1:N-1;
```

```
        R[k]=euler(phi[k],theta[k],psi[k]);
```

```
    end
```

```
end
```

```
//-----
```

```
contractor-list statequ
```

```
    for k=1:N-1;
```

```
        p[k+1]=p[k]+0.1*R[k]*v[k];
```

```
    end
```

```
end
```

```
//-----
```

```
contractor init
```

```
    inter k=1:N-1;
```

```
        rotation(k)
```

```
    end
```

```
end
```

```
contractor fwd
```

```
    inter k=1:N-1;
```

```
        statequ(k)
```

```
    end
```

```
end
```

```
//-----
```

```
contractor bwd
```

```
    inter k=1:N-1;
```

```
        statequ(N-k)
```

```
    end
```

```
end
```

```
main
    p[1] :=read("gps_init.dat");
    v :=read("Quimper_v.dat");
    phi :=read("Quimper_phi.dat");
    theta :=read("Quimper_theta.dat");
    psi :=read("Quimper_psi.dat");
    init;
    fwd;
    bwd;
    column(p,px,1);
    column(p,py,2);
    print("--- Robot positions: ---");
    newplot("gesmi.dat");
    plot(px,py,color(rgb(1,1,1),rgb(0,0,0)));
end
```

5 Sailboat robotics

5.1 Vaimos



Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) .$$

With the controller $\mathbf{u} = \mathbf{g}(\mathbf{x})$, the robot satisfies an equation of the form

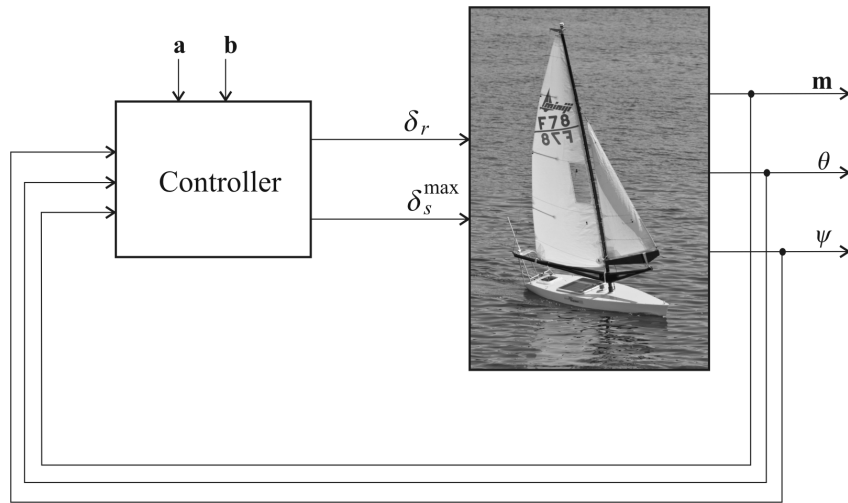
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) .$$

With all uncertainties, the robot satisfies.

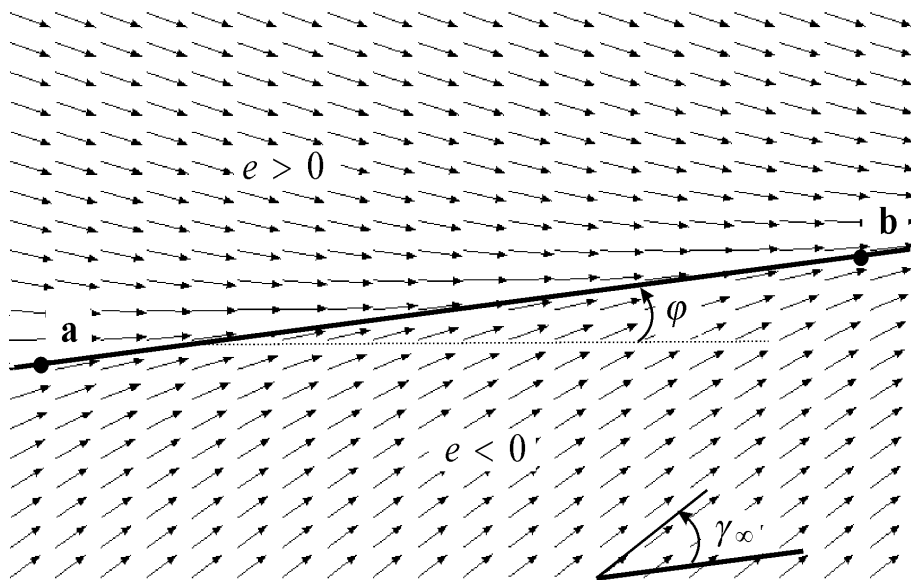
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

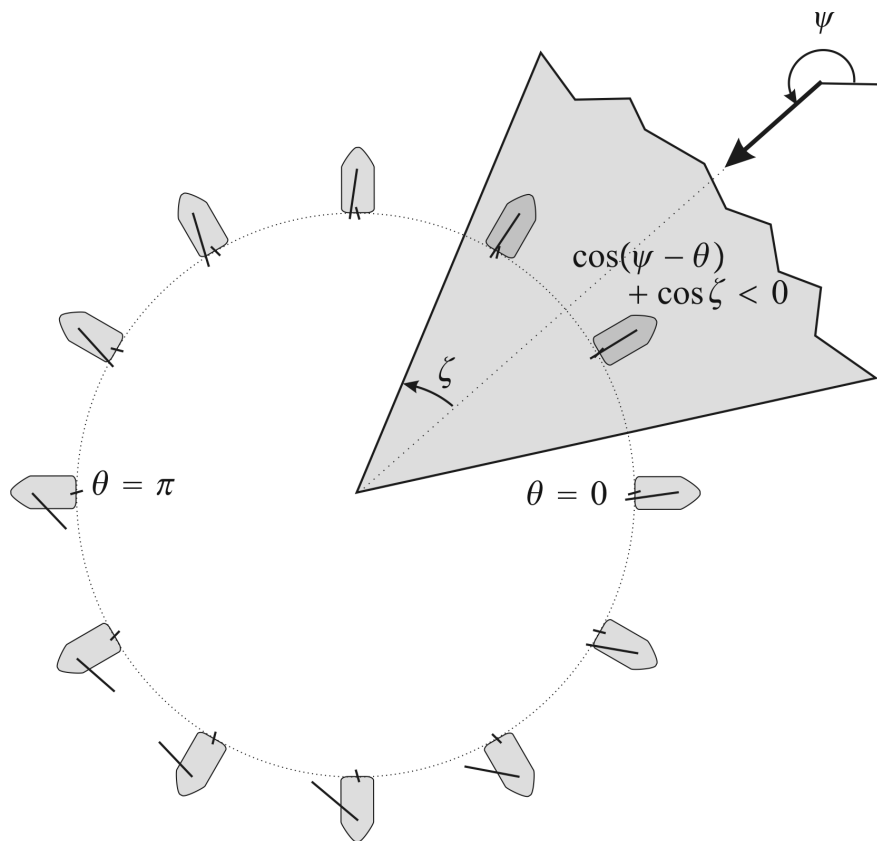
5.2 Line following

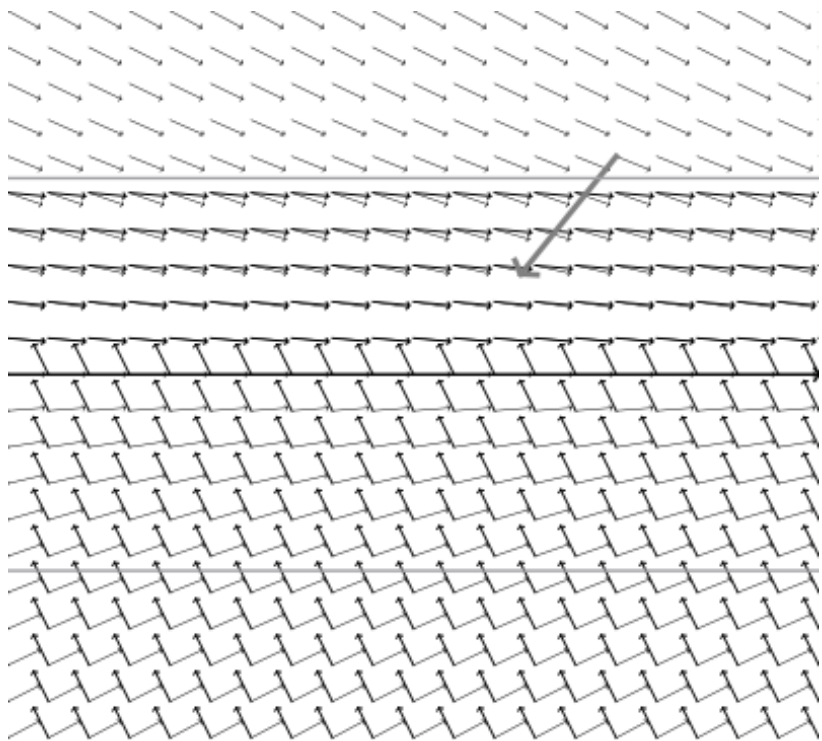


Controller of a sailboat robot



Nominal vector field θ^*





Keep close hauled strategy

5.3 V -stability

The system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

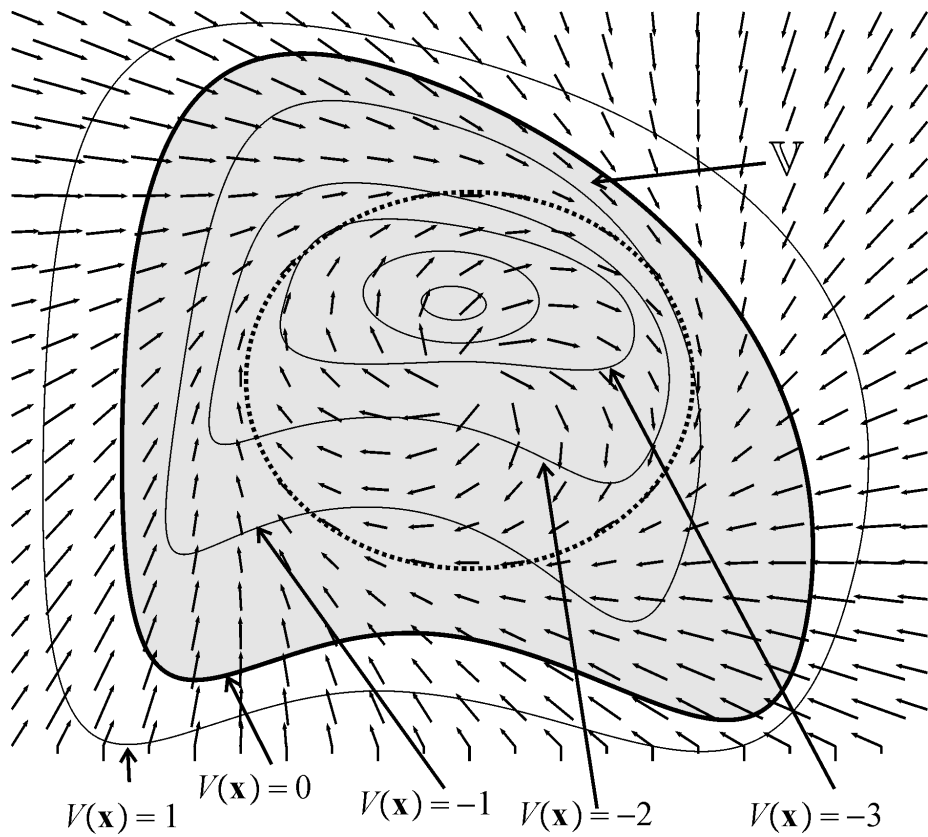
is Lyapunov-stable (1892) if there exists $V(\mathbf{x}) \geq 0$ such that

$$\dot{V}(\mathbf{x}) < 0 \text{ if } \mathbf{x} \neq \mathbf{0}$$

$$V(\mathbf{x}) = 0 \text{ iff } \mathbf{x} = \mathbf{0}.$$

Definition. Consider a differentiable function $V(\mathbf{x})$. The system is V -stable if we have

$$\dot{V}(\mathbf{x}) < 0 \text{ if } V(\mathbf{x}) \geq 0.$$



Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V -stable then

- (i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
- (ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0, V(\mathbf{x}(t + \tau)) < 0$.

Now,

$$\begin{aligned} & \left(V(\mathbf{x}) \geq 0 \Rightarrow \dot{V}(\mathbf{x}) < 0 \right) \\ \Leftrightarrow & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0 \right) \\ \Leftrightarrow & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\ \Leftrightarrow & \neg \left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \geq 0 \text{ and } V(\mathbf{x}) \geq 0 \right) \end{aligned}$$

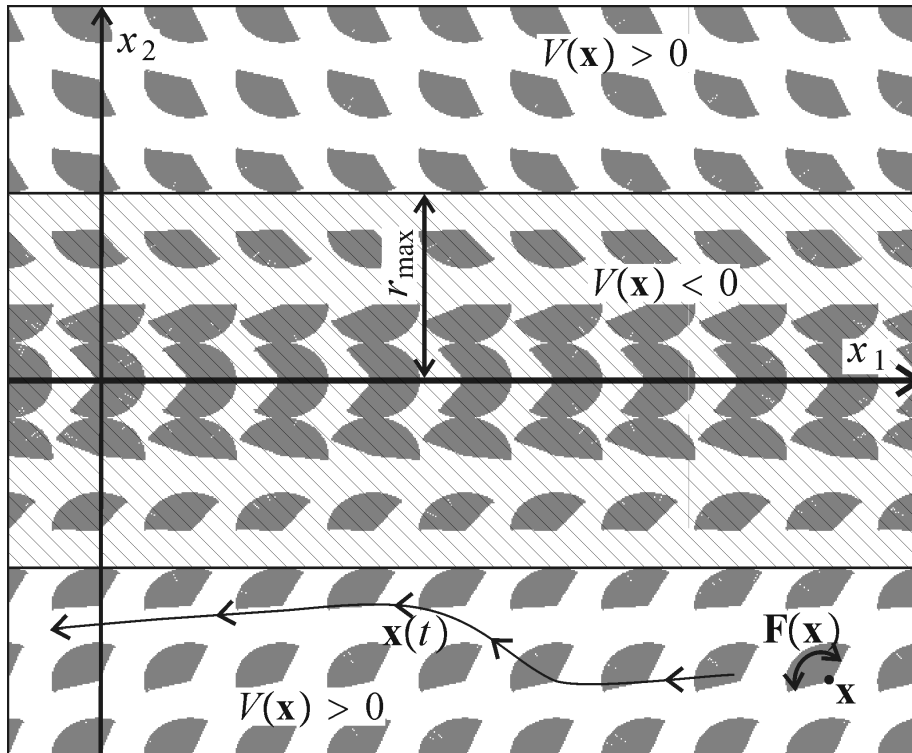
Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent} \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V -stability.

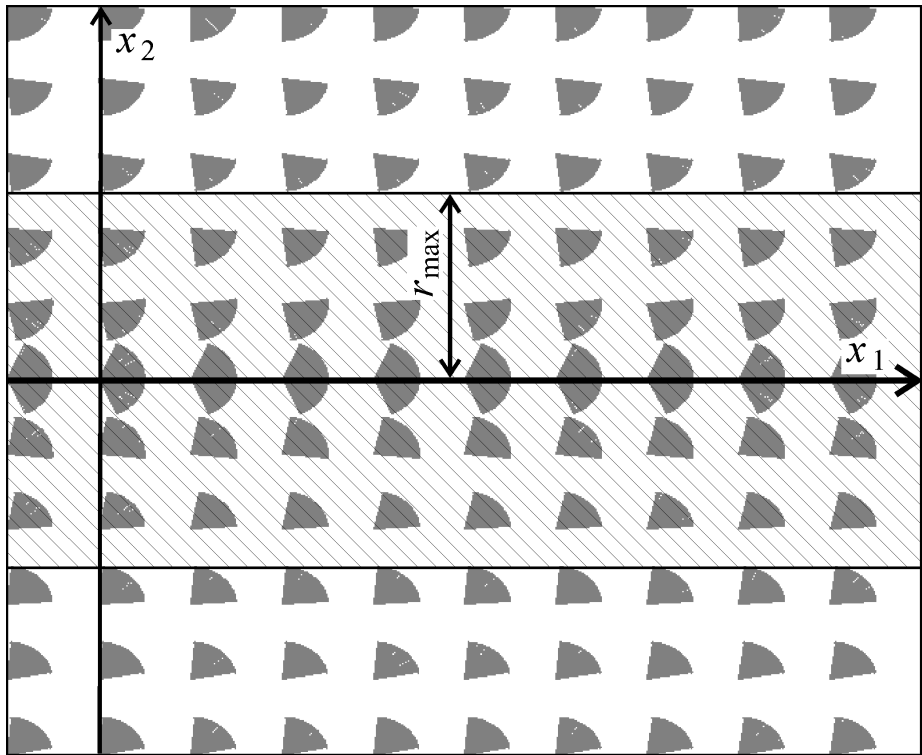
Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ V(\mathbf{x}) \geq 0 \end{array} \right. \quad \text{inconsistent} \Leftrightarrow \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$

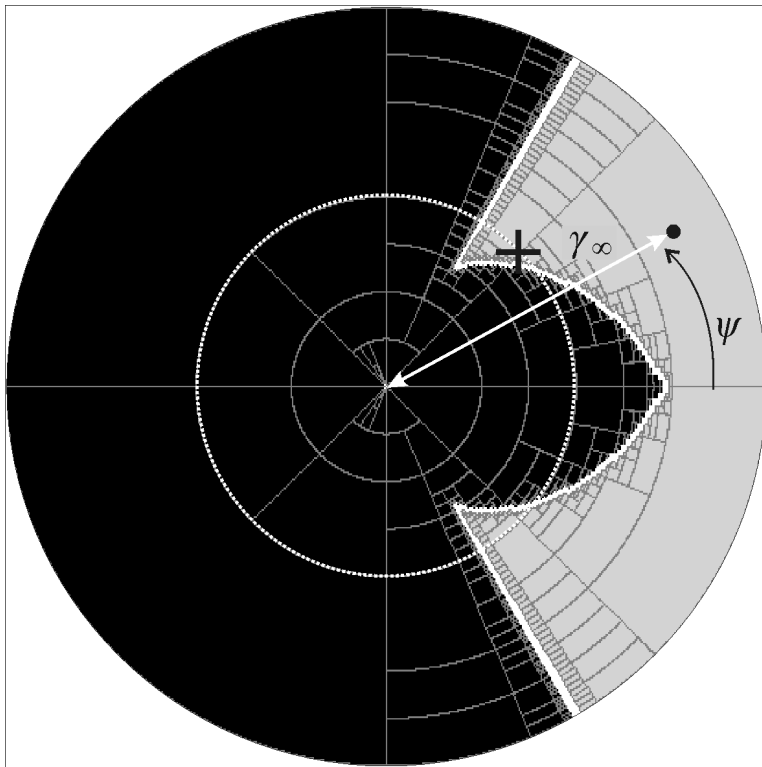


5.4 Parametric case

Consider the differential inclusion

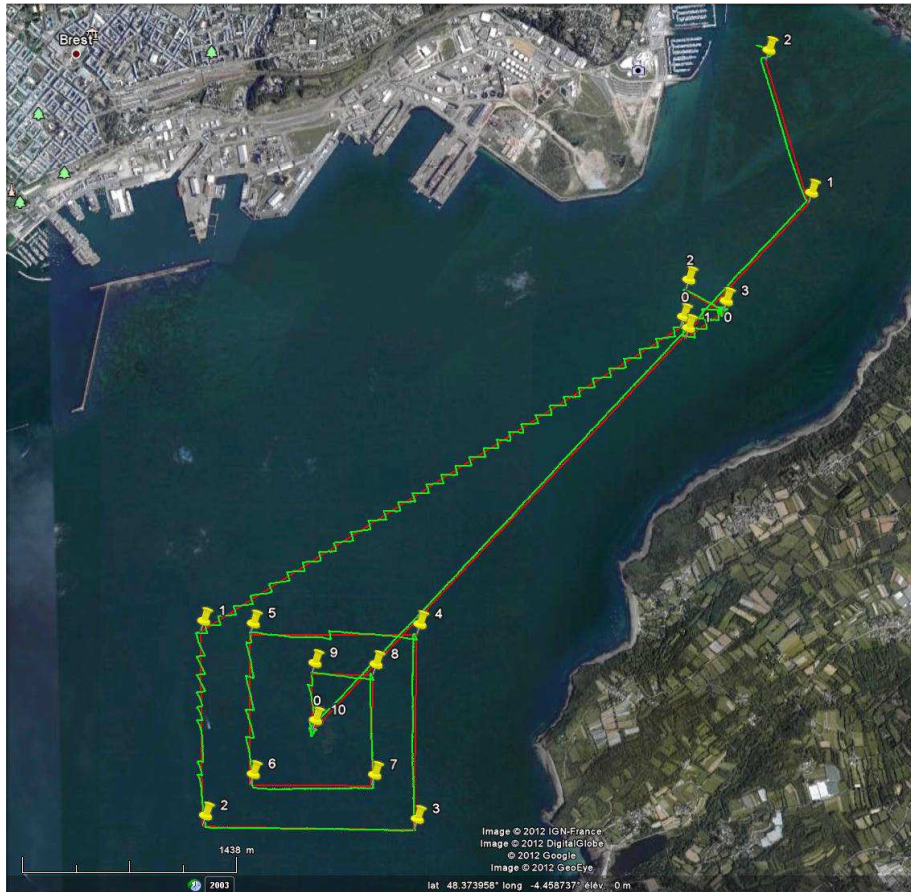
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V -stable.



5.5 Experimental validation

Brest



Brest-Douarnenez. January 17, 2012, 8am

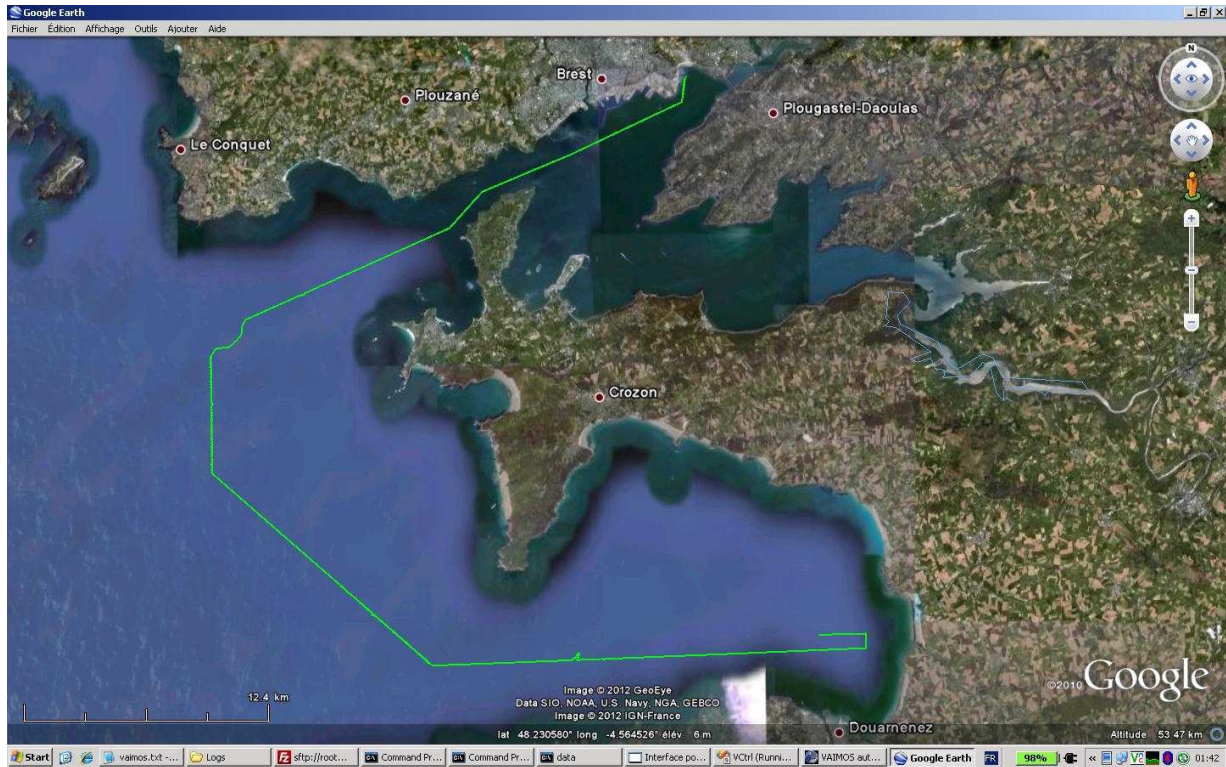




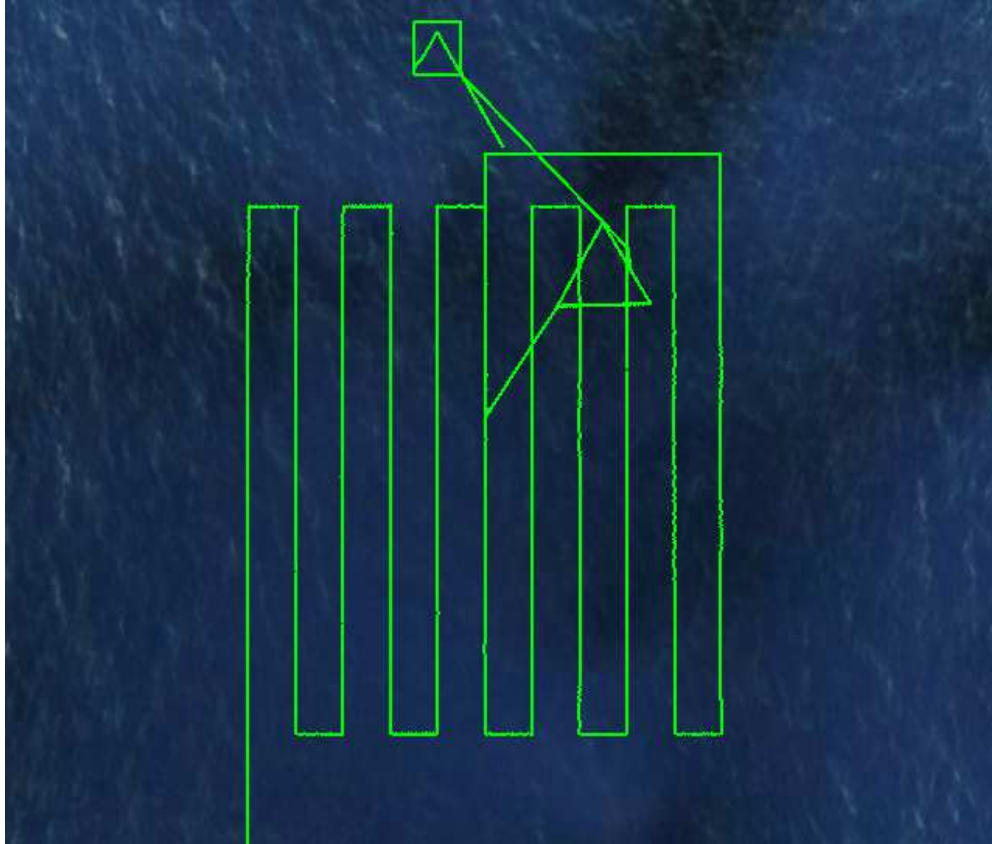








Middle of Atlantic ocean



350 km made by Vaimos in 53h, September 6-9, 2012.

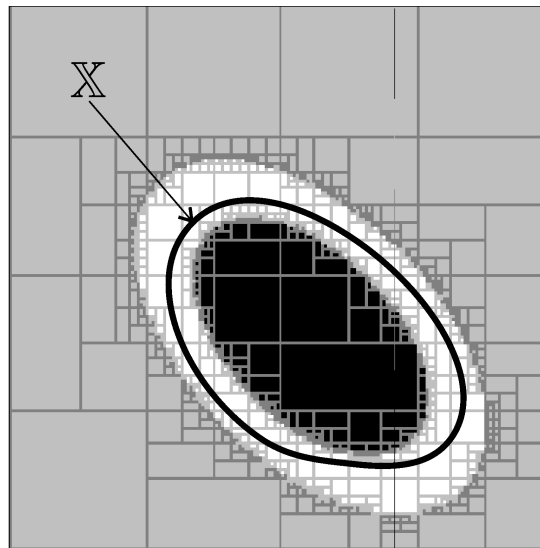
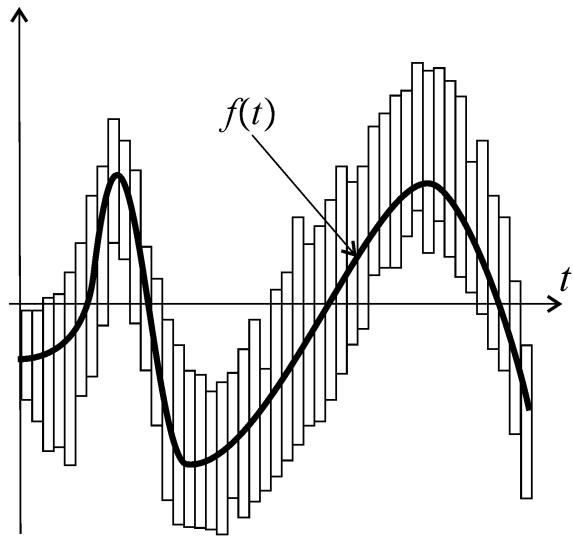
Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

6 Perspectives



An interval function (or tube) and a set interval

