Computer-assisted error analysis for second-order elliptic equations in divergence form

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Considered Problems $(N = 1, 2, 3)$

$\Omega \subset \mathbb{R}^N$: bounded domain with Lipschitz boundary

$$(P_N) \quad \begin{cases} -\mathrm{div}\ (a(x)\nabla u) = f(x), & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Assuming

- $a(x) \in W^{1,\infty}(\Omega)$, non-negative function.
- $f(x) \in L^2(\Omega)$.
- For $N = 1$, the problem is

$$(P_1) \quad \begin{cases} -(a(x)u')' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

(or $u(0) = u'(1) = 0$ on boundary).

Existence of Solution

+

(Computable) Error estimate

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Weak formulation

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Note

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- $f(x) \in L^2(\Omega)$.

Weak formulation

$$(P_N) \Longrightarrow \text{Find } u \in V, \quad a(u,v) = (f,v), \quad \forall v \in V.$$

Note

- $V = H_0^1(\Omega)$
- $a(u,v) := (a(x)\nabla u, \nabla v)$.

Weak formulation

$$(P_N) \Longrightarrow \text{Find } u \in V, \quad (\mathcal{A}u, v)_V = (f, v), \quad \forall v \in V.$$

Note

- $\mathcal{A} : V \to V$, $(\mathcal{A}u, v)_V := a(u, v)$,
- $\exists w_f \in V : (w_f, v)_V := (f, v)$, $\longleftarrow$ Riesz's rep. th.

Weak formulation

$$(P_N) \Longrightarrow \text{Find } u \in V, \quad (\mathcal{A}u, v)_V = (w_f, v)_V, \quad \forall v \in V.$$

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Weak formulation

$$(P_N) \Longrightarrow \text{Find } u \in V \text{ satisfying } \mathcal{A}u = w_f \text{ in } V.$$

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Existence of weak solution

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Lax-Milgram theorem

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Continuity & Coercivity

Let $a(\cdot,\cdot) : V \times V \rightarrow \mathbb{R}$ be a bilinear form.

Continuity & Coercivity

For $M, \lambda > 0$,

$$\begin{aligned} |a(u,v)| &\leq M\|u\|_V\|v\|_V, \\ a(u,u) &\geq \lambda\|u\|_V^2. \end{aligned}$$

e.g. for $\Omega = (0,1)$,

$$a(x) = 1 + x^2, \quad M = 1.$$

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$$a(x) = 1 - x^2, \quad M = 1.$$

Discretization

Let

$$V_h := \text{span}\{\phi_1, ..., \phi_n\} \subset V,$$

be finite element subspace spanned by base functions $\{\phi_i\}$.

Let $\mathcal{P}_h : V \to V_h$ be the orthogonal projection defined by

$$(\nabla(u - \mathcal{P}_h u), \nabla v_h) = 0, \quad \forall v_h \in V_h.$$

Let $\mathcal{R}_h : V \to V_h$ be the orthogonal projection defined by

$$a\left(u - \mathcal{R}_h u, v_h\right) = 0, \quad \forall v_h \in V_h.$$

Facts for Poisson's equation

Starter: Poisson's eq.

Note that following facts:

$$-\Delta u = g \text{ with } u|_{\partial\Omega} = 0 \Rightarrow \mathcal{L}u = w_g \text{ in } V,$$

where

$$(\mathcal{L}u, v)_V := (\nabla u, \nabla v), \ \ (w_g, v)_V := (g, v), \quad \forall v \in V.$$

From Riesz's representation theorem, $\exists u = \mathcal{L}^{-1} w_g$ s.t.

$$\begin{aligned} \|u - \mathcal{P}_h u\|_V &= \|(\mathcal{I} - \mathcal{P}_h)\mathcal{L}^{-1} w_g\|_V \\ &\leq C(h)\|g\|_{L^2}. \end{aligned}$$

$$C(h) := \sup_{0 \neq g \in L^2(\Omega)} \frac{\|(\mathcal{I} - \mathcal{P}_h)\mathcal{L}^{-1} w_g\|_V}{\|g\|_{L^2}} \approx O(h^{1-\varepsilon}).$$

Error estimate (P_1-element)

Convex domain

$$C(h) \le \sup_{0 \ne u \in H^2 \cap H_0^1} \frac{\|\nabla(u - \mathcal{P}_h u)\|_{L^2}}{|u|_{H^2}}$$

is evaluated by interpolation error in Nakao et. al. (1998), Kikuchi & Lilu (2007), Kobayashi (2010).

Non-convex domain

$$C(h) \le \sqrt{C_{0,h}^2 + \kappa_h^2}$$

Liu & Oishi (2010) calculated using *Hyper-circle* approach:

$$\|u - \mathcal{P}_{0,h} u\|_{L^2} \le C_{0,h} \|\nabla u\|_{L^2}.$$

Let $\mathcal{P}_{0,h} u$ be piecewise constant mapping of u and κ_h computable value.

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Error estimate (P_2-element)

Arbitrary polygonal domain

$$C(h) \le \sqrt{C_{1,h}^2 + \kappa_h^2}$$

T., Liu & Oishi (2012) calculated using *Hyper-circle* approach:

$$\|u - \mathcal{P}_{1,h}u\|_{L^2} \le C_{1,h}\|\nabla u\|_{L^2}.$$

Let $\mathcal{P}_{1,h}u$ be piecewise discontinuous linear mapping of u and κ_h computable value.

How about $-\mathrm{div}(a\nabla u) = f$?

Divergence form

The weak formulation revisited.

$$-\mathrm{div}(a\nabla u) = f \text{ with } u|_{\partial\Omega} = 0 \Rightarrow \mathcal{A}u = w_f \text{ in } V,$$

If the ellipticity (Continuous & Coercivity) is obtained, then Lax-Milgram's theorem states $\exists u = \mathcal{A}^{-1}w_f$ s.t.

$$\begin{aligned} \|u - \mathcal{R}_h u\|_V &= \|(\mathcal{I} - \mathcal{R}_h)\mathcal{A}^{-1}w_f\|_V \\ &\leq C_a(h)\|f\|_{L^2}. \end{aligned}$$

$$C_a(h) := \sup_{0\neq f\in L^2(\Omega)} \frac{\|(\mathcal{I} - \mathcal{R}_h)\mathcal{A}^{-1}w_f\|_V}{\|f\|_{L^2}} \approx O(h^{1-\varepsilon}) \ ?$$

ellipticity case Céa's lemma works:

For approximation $\mathcal{R}_h u \in V_h$, satisfying

$$a\left(\mathcal{R}_h u, v_h\right) = (f, v_h), \quad \forall v_h \in V_h.$$

It follows

$$\begin{aligned}
\|u - \mathcal{R}_h u\|_V &\leq \frac{M}{\lambda} \inf_{w_h \in V_h} \|u - w_h\|_V \leftarrow \text{Céa's lemma} \\
&\leq \frac{M}{\lambda} \|u - \mathcal{P}_h u\|_V \\
&\leq \frac{M}{\lambda} C(h) \|\Delta u\|_{L^2} \\
&\leq \frac{M}{\lambda} C(h) C' \|f\|_{L^2} \approx O(h^{1-\varepsilon}).
\end{aligned}$$

Degenerate case Using discrete infsup (LBB-)condition

$$\|u - \mathcal{R}_h u\|_V \quad \leq \quad \frac{M}{C_h} C(h) \tilde{C} \|f\|_{L^2},$$

where

$$C_h := \inf_{0 \neq u_h \in V_h} \sup_{0 \neq v_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} \approx O(h^{-1}).$$

Then, $\|u - \mathcal{R}_h u\|_V \approx O(1 \text{ or } h^{-1})$! That's too bad...

Setting of the function space is not suitable.

Theorem (Existence of solution)

$\mathcal{A}: V \to V$ is continuous with $\|\mathcal{A}\|_{V,V} \le M$. K satisfies,

$$\|(\mathcal{L}-\mathcal{A})u_c\|_{L^2} \le K\|u_c\|_V,$$

for $\forall u_c = u - \mathcal{P}_h u \in V_c (= V \setminus V_h)$. $\mathcal{A}_h : V_h \to V_h$ is invertible with

$$\left\|\mathcal{A}_h^{-1}\right\|_{V,V} \le C_h^{-1}.$$

Moreover, error estimate of Poisson's equation is obtained for given $g \in L^2(\Omega)$:

$$\|u - \mathcal{P}_h u\|_V \le C(h)\|g\|_{L^2}.$$

If $\nu_h := 1 - C(h)C_{e,2}(C_h^{-1}M^2 + K) > 0$,

such that $\mathcal{A}: V \to V$ is invertible.

Model problem

$$\begin{cases} -(a(x)u')' = f(x), & 0 < x < 1, \\ u(0) = 0, \ u'(1) = 0. \end{cases}$$

$a(x) \in W^{1,\infty}(\Omega)$,

$$f(x) = 4\pi^2(1 + x^2)\sin(2\pi x) - 4\pi x \cos(2\pi x)$$

Let us define $V := \{v \in H^1(\Omega) \ : \ v(0) = 0\}$ with $(\cdot, \cdot)_V$.

Carried out on

Mac OS X 10.6.8, 2.7 GHz CPU, MATLAB 2011a, Assembling matrix and vector by symbolic math toolbox (MuPAD)

Results

$a(x) = 1 - x^2,\ M = 1$

$2^{-\eta}$	$C(h)$	C_h	$\nu_h(> 0)$	Existence
3	1.98944×10^{-2}	5.16106×10^{-2}	0.87327	OK
4	9.94718×10^{-3}	2.61107×10^{-2}	0.87672	OK
5	4.97359×10^{-3}	1.31316×10^{-2}	0.87843	OK
6	2.48680×10^{-3}	6.58486×10^{-3}	0.87929	OK
7	1.24340×10^{-3}	3.29719×10^{-3}	0.87971	OK
8	6.21699×10^{-4}	1.64979×10^{-3}	0.87992	OK
9	3.10849×10^{-4}	8.25190×10^{-4}	0.88003	OK

Results

$a(x) = (1-x)^2,\ M = 1$

$2^{-\eta}$	$C(h)$	C_h	$\nu_h(>0)$	Existence
3	1.98944×10^{-2}	1.18865×10^{-3}	-4.33158	NO
4	9.94718×10^{-3}	2.97162×10^{-4}	-9.65711	NO
5	4.97359×10^{-3}	7.42904×10^{-5}	-20.3112	NO
6	2.48680×10^{-3}	1.85726×10^{-5}	-41.6208	NO
7	1.24340×10^{-3}	4.64315×10^{-6}	-84.2410	NO
8	6.21699×10^{-4}	1.16079×10^{-6}	-169.481	NO
9	3.10849×10^{-4}	2.90197×10^{-7}	-339.963	NO

Weighted Sobolev space

[1] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem. Nonlinear differ. equ. appl. 7 (2000) 187–199.

Assumption on $a(x)$

Let $a \in L^1_{\text{loc}}(\Omega)$ be a given function on Ω. We suppose (h_α) for $\alpha \in (0, 2]$,

$$\liminf_{x \to z} \frac{a(x)}{|x - z|^\alpha} > 0, \quad \forall z \in \overline{\Omega}. \qquad (h_\alpha)$$

Let us introduce the value $2^*_\alpha = 4/\alpha \in [2, \infty)$.

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$$V_a := \{v \in H_0^1(\Omega) : x \in \text{supp } a(x)\} \subset V$$

with its inner product and the norm for $\alpha \in (0, 2]$

$$(u, v)_a = a(u, v), \quad \|u\|_a = a(u, u)^{1/2}, \quad \text{for } u, v \in V_a.$$

Results of embeddings (Lem. 3.1 & Prop. 3.2 in [1])

Let Ω be bounded and $a \in L^1_{\text{loc}}(\Omega)$ satisfying (h_α) for $\alpha \in (0, 2]$. The following embeddings hold

- $V_a \hookrightarrow L^{2^*_\alpha}(\Omega)$ continuously
- $V_a \hookrightarrow L^p(\Omega)$ with compact inclusion if $p \in [1, 2^*_\alpha)$

cf. $\alpha \in (1, 2)$ in ODE case.

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Future work (A priori error estimate)

Our goal of this study is a priori error estimate for degenerate ellipticity problem $a(x) \geq 0$:

$$\|u - \mathcal{R}_h u\|_a \quad \leq \quad C_a(h)\|f\|_{L^2},$$

where

$$C_a(h) := \sup_{0 \neq f \in L^2(\Omega)} \frac{\|(\mathcal{I} - \mathcal{R}_h)\mathcal{A}^{-1} w_f\|_a}{\|f\|_{L^2}} \approx O(h^{1-\varepsilon}) \ ?$$

Thank you for kind attention !