Klein-Gordon Equation and Differential Substitutions

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This paper is devoted to classification problem of nonlinear hyperbolic equations of the form

$$u_{xy} = f(u, u_x, u_y)\,. \qquad (1)$$

These equations have numerous applications to physical problems (continuum mechanics, quantum field theory, the theory of ferromagnetic).

It is known that the hallmark of integrability of the equation is the existence of higher symmetries. However, the symmetry classification method has proved very effective only for evolution equations. As for nonlinear hyperbolic equations, historically the first complete list of equations of special type

$$v_{xy} = F(v) \qquad (2)$$

Klein-Gordon equations possessing higher symmetries was obtained in [1]. Application of the symmetry approach to hyperbolic equations of general form (1) facing special difficulties.

In this paper we present a classification based on differential substitutions.

Definition. The relation

$$v = \Phi\left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial x^n}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^m u}{\partial y^m}\right) \qquad 3)$$

is called differential substitution from equation (1) into equation

$$v_{xy} = g(v, v_x, v_y)\,, \qquad (4)$$

if function (3)satisfies equation (4) for every solution of equation (1).

The relationship between the solutions of these equations, in general, irreversible, and is one-sided. However, the differential substitutions can be consequences of Backlund transformations[2].

Nowadays you can find (see [3]) a lot of examples of equations of the form (1) reducible by differential substitutions to known integrable models. Thus, the solution of equation

$$u_{xy} = s(u)\sqrt{1-u_x^2}\sqrt{1-u_y^2}$$

under the transformation $v = \arcsin u_x + \arcsin u_y + p(u)$ becomes to the solution of sine-Gordon equation integrable by the inverse scattering problem. Here $s'' - 2s^3 + \lambda s = 0$, λ is an arbitrary constant, $p'^2 = 2s' - 2s^2 + \lambda$. Constant c is determined by the value of the first integral $I = s'^2 - s^4 + \lambda s^2$ by the formula $I = c^2$.

The purpose of this paper is to describe all the nonlinear hyperbolic equations of the form (1), which reduced to Klein-Gordon equation (2) by differential substitutions

$$v = \varphi(u, u_x, u_y) \,. \qquad (5)$$

In other words, the problem is to determine functions f, F and φ. As a result, we find all the equations (1), (2) and transformation (5), connecting their solutions. The inverse problem, which is a description of all equations of the form (2), reducible in equation (1) by substitutions $u = \psi(v, v_x, v_y)$ is also solved.

References

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