Solitons, Collapses and Self-Similar Solutions in Cahn-Hilliard Kind Equation

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We study the qualitative properties of the solutions of the equation of Cahn-Hilliard kind

$$u_t + \Delta^2 u + \Delta(u^2 - \beta u) = 0; \quad u = u_0(x, y), \; t = 0, \qquad (1)$$

where β is a constant. The equation (1) has arisen from analyses of the thermocapillary flow in the thin layer of viscous liquid with a free surface at non-monotonic dependence of the surface tension σ on the temperature θ. In some solutions, melted steel and alloys the dependence is well described by the equation $\sigma = \sigma_0 + \phi(\theta - \theta_*)^2$ with appropriate positive constants σ_0, θ_*, ϕ. Anomalous Marangoni effect in two-layer system was investigated on the basis of the full Navier-Stokes equations in [2]. In the thin layer approximation we have investigated the Rayleigh-Benard problem with the condition $\theta = \theta_*$ on the free surface. We assume that the characteristic disturbance amplitude of free surface $u(x, y, t)$ is much less than the average layer thickness. In this case the evolution of the non-dimensional deviation of a free boundary from a horizontal equilibrium state can be described in terms of Cauchy problem solutions for the equation (1), and β is the Bond number. The sufficient condition of the global solution existence of problem (1) and its collapse for a finite time for the periodic function u_0 has been formulated. Realization of both possibilities at rapid decreasing of u_0 at infinity in two-dimensional and axially symmetric cases is numerically modeled.

The equation (1) is reduced to the case $\beta = 0$ by substitution $u = u' + \beta/2$. In this case the equation (1) has self-similar solutions. The ”mass” conservation law takes place for Cauchy problem (1)

$$\int_{\mathbb{R}^2} u\,dx\,dy = \int_{\mathbb{R}^2} u_0\,dx\,dy = c, \qquad (2)$$

quotes are caused by that the value c can be negative. Space-periodic solutions of Cauchy problem and rapidly decreasing solutions at infinity are studied. The behavior of Cauchy problem solutions (1) are following: or $u \to u_s$ when $t \to \infty$, where u_s is some stationary solution, or its solution is destroyed for a infinite or infinite time. Analytical and numerical research shows that axially symmetric self-similar solutions exist at small values of $|c|$, and they do not exist for large and positive c. For negative values of c there were found two branches of self-similar solutions with various qualitative behaviors. Such solutions form one-parameter family with the parameter c. The self-similar solutions of the plane problem satisfying the conservation law exist only for $c = 0$. The role of self-similar solutions is that they often give the leading term of asymptotic solution of Cauchy problem (1) as $t \to \infty$. In our case it is so for axially symmetric solutions, where $u = O(t^{-1/2})$. For the self-similar solution of two-dimensional problem at $c = 0$ an order of decrease is the same as in the axially symmetric case, while the Cauchy problem solution with small initial data has a decrease order $t^{-1/4}$ as $t \to \infty$.

The equation (1) has numerous stationary solutions. A special place occupies solutions, which depends on one variable only. In this case the equation is integrated in elliptic functions, the corresponding periodic solutions are the well-known cnoidal waves. Another interesting set of stationary solutions consists of solutions rapidly decreasing when one or both variables tend to infinity. As an example of such solution it is Korteweg and de Vries solitons. There are axially symmetric solitons among the stationary solutions. Solutions having a “large” initial norm can be destroyed for a finite time. The collapse takes place for a simple example of an initial function of Cauchy problem $u_0 = a_1 \cos x + a_2 \cos 2x$, where constants a_1 and a_2 submit to inequalities $|a_1| > 2$, $a_1^2 - (a_1^4 - 16)^{1/2} < 2a_2 < a_1^2 + (a_1^4 - 16)^{1/2}$.

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References

[1] J.C. Legros, M.C. Limbourg-Fontaine and G. Petre, “Influence of a surface tension minimum as a function of temperature on the Marangoni convection”, Acta Astronautica **11**, 143-147 (1984).

[2] T. Boeck, A. Nepomnyashchy, I. Simanovskii, A. Golovin, L. Braverman and A. Thess, “Three-dimensional convection in a two-layer system with anomalous thermocapillary effect”, Physics of Fluids **14**, 3899-3911 (2002).