Self-Induced Transparency (SIT) in a Dispersive Medium

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Abstract

We present physical onsets of novel integrable generalizations of the Maxwell-Bloch equations describing electromagnetic field interaction with a two-level systems (TLS).

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Self-induced-transparency (SIT) soliton phenomenon in two-level atomic systems is one of the most well-known coherent pulse propagation phenomena.

S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967).

2. Integrable Maxwell-Bloch equations (MBE) in a two-level system (TLS) with slow varying envelope approximation.

G. L. Lamb. Jr, Rev. Mod. Phys. **43**, 99 (1971).

3. Integrable generalizations of the MBEs in TLS.

A. A. Zabolotskii, Phys. Lett. A **124**, 500 (1987).(Nonlinear Stark shift).

M. Agrotis, N.M. Ercolani, S.A. Glasgow, J.V. Moloney, Physica D, **138** 134 (2000).(Permanent dipole momentum)

A.A. Zabolotskii, JETP Lett. **77**, 464 (2003).(Two polarizations)

H. Steudel, A.A. Zabolotskii, R. Meinel, Phys. Rev. E **72**, 056608 (2005).

A.A. Zabolotskii, Phys. Rev. E **77**, 036603 (2008).(Two polarizations + permanent dipole).

+ N-level systems, + unified models (Nonlinear Schr odinder + Maxwell-Bloch).

I. DISPERSIVE HOST MEDIUM

The Maxwell equation is

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi N}{c^2}\frac{\partial^2 P_{\rm nl}}{\partial t^2}, \tag{1}$$

here N is a density of the TLS. Nonlinear part of the polarizability is

$$P_{\rm nl}(x,t) = \int_0^t \epsilon(t-t')P_{TLS}(x,t')dt' \tag{2}$$

The susceptibility $\epsilon(t)$ describes the retarded reaction. For a TLS $P_{TLS} = (d_{21}\rho_{12} + d_{12}\rho_{21})$, where $d_{12} = d_{21}^*$ are the elements of the dipole matrix d_{ij}. $\rho_{ij},\ i,j = 1,2$ is the density matrix of the TLS.

Dielectric medium

$$\epsilon_{Lorentz}(\omega) = \epsilon_0\left[\epsilon_\infty + \frac{\Delta\epsilon_p\omega_p^2}{(\omega_p^2 - \omega^2 - 2i\Gamma_p\omega)}\right], \tag{3}$$

Present the electromagnetic field amplitude as

$$E(x,t) = \frac{1}{2}\left[\mathcal{E}(x,t)e^{i(k_0x-\omega_0t)} + \mathcal{E}^*(x,t)e^{-i(k_0x-\omega_0t)}\right], \tag{4}$$

where $\pm\omega_0$ and $\pm k_0$ are the carrying frequencies and the wave vectors, respectively, $\omega_0 = ck_0$. $\mathcal{E}(x,t)$ is the slow envelope:

$$\left|\frac{\partial^2\mathcal{E}}{\partial t^2}\right| \ll \omega_0\left|\frac{\partial\mathcal{E}}{\partial t}\right|, \tag{5}$$

$$\left|\frac{\partial^2\mathcal{E}}{\partial x^2}\right| \ll k_0\left|\frac{\partial\mathcal{E}}{\partial x}\right|. \tag{6}$$

Let

$$P_{TLS}(x,t) = \frac{d_{12}}{2}\left[S(x,t)e^{i(k_0x-\omega_0t)} + S(x,t)^* e^{-i(k_0x-\omega_0t)}\right], \quad (7)$$

$S(x,t)$ is the slow amplitude of off-diagonal elements of the density matrix $\rho_{12}(x,t)$:

$$\rho_{12}(x,t) = S(x,t)e^{i(k_0x-\omega_0t)},\ \rho_{21}(x,t) = S^*(x,t)e^{-i(k_0x-\omega_0t)}. \quad (8)$$

Neglect the terms $(|\partial_t S|/\omega_0)^n, n = 1,2,$ $(|\partial_t \epsilon(t)|/\omega_0)^n$ and $(|\partial_t S(t)|/\omega_0)^n, n = 1,2.$

Let

$$\epsilon(\omega) = \epsilon(\omega_0) + \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + \frac{1}{2}\left.\frac{\partial^2 \epsilon}{\partial \omega^2}\right|_{\omega=\omega_0}(\omega-\omega_0)^2 + \cdots . \quad (9)$$

Then

$$P_{\mathrm{nl}}(x,t) = \mathrm{e}^{i(k_0x-\omega_0t)}\sum_k \frac{1}{k!}\left.\frac{\partial^k \epsilon}{\partial \omega^k}\right|_{\omega=\omega_0}\left(i\frac{\partial}{\partial t}\right)^k d_{21}S(x,t)(x,t)$$
$$+\mathrm{e}^{-(k_0x-\omega_0t)}\sum_k \frac{1}{k!}\left.\frac{\partial^k \epsilon}{\partial \omega^k}\right|_{\omega=-\omega_0}\left(i\frac{\partial}{\partial t}\right)^k d_{21}S^*(x,t)(x,t). \quad (10)$$

From Maxwell equations we get

$$ie^{i(k_0x-\omega_0t)}\left(\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{E}}{\partial x}\right) = -\frac{2\pi N\omega_0^2 d_{12}}{c^2}$$
$$\times\left[e^{i(k_0x-\omega_0t)}\sum_k i^k q_k \left(\frac{\partial}{\partial t}\right)^k S(x,t)\right], \quad (11)$$

where

$$q_k = \frac{1}{k!}\left.\frac{\partial^k \epsilon(\omega)}{\partial \omega^k}\right|_{\omega=\omega_0}. \quad (12)$$

Maxwell equation, neglecting all terms with $k > 2$,

$$\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{E}}{\partial x} = \frac{2\pi\omega_0^2 N d_{21}}{k_0 c^2}\left(iq_0 S - q_1\frac{\partial S}{\partial t} - iq_2\frac{\partial^2 S}{\partial t^2}\right). \quad (13)$$

The Bloch equations are:

$$\partial_t \rho_{12} = -i\omega_{12}\rho_{12} - i\left(\rho_{11} - \rho_{22}\right)\frac{d_{12}}{\hbar}E, \quad (14)$$

$$\partial_t \rho_{11} = i\frac{d_{12}}{\hbar}E\left(\rho_{21} - \rho_{12}\right), \quad (15)$$

$$\partial_t \rho_{22} = i\frac{d_{12}}{\hbar}E\left(\rho_{12} - \rho_{21}\right), \quad (16)$$

here ω_{12} is a frequency of the two-level transition.

Novel integrable dispersive Maxwell-Bloch equations (DMBEs):

$$\frac{\partial S}{\partial \tau} = i\nu S - i\mathcal{U} S_z, \quad (17)$$

$$\frac{\partial S_z}{\partial \tau} = i\left(\mathcal{U} S^* - \mathcal{U}^* S\right), \quad (18)$$

$$\frac{\partial \mathcal{U}}{\partial \chi} = ir_0 S + r_1\frac{\partial S}{\partial \tau} + ir_2\frac{\partial^2 S}{\partial \tau^2}, \quad (19)$$

here $\tau = (t - x/c)\omega_R$, $\nu = (\omega_0 - \omega_{12})/\omega_R$, $S_z = \rho_{11} - \rho_{22}$, and

$$\mathcal{U} = \frac{d_{12}\mathcal{E}}{\hbar\omega_R}, \quad (20)$$

$$r_0 = q_0,\ r_1 = -q_1\omega_R,\ r_2 = -q_2\omega_R^2, \quad (21)$$

$$\frac{\partial}{\partial \chi} = \frac{c\hbar\omega_R}{2\pi d_{12}^2\omega_0 N}\frac{\partial}{\partial x}. \quad (22)$$

II. A ZERO CURVATURE PRESENTATION

$r_0, r_1, r_2, \nu \in \mathbb{R}$, $r_2 \neq 0, r_0 - \nu r_1 - \nu^2 r_2 \neq 0$.

$$\partial_\tau \Phi = \begin{pmatrix} -i\lambda & m_0\left(\lambda + b_-\right)U \\ m_0\left(\lambda + b_+\right)U^* & i\lambda \end{pmatrix} \Phi, \tag{23}$$

$$\partial_\chi \Phi = \mathbf{A}\Phi \equiv \begin{pmatrix} i\alpha_0 S_z & \mu\left(\alpha_1 S + i r_2 \partial_\tau S\right) \\ \widetilde{\mu}\left(\alpha_1 S^* - i r_2 \partial_\tau S^*\right) & -i\alpha_0 S_z \end{pmatrix} \Phi, \tag{24}$$

$$\mu = m_0\left(\lambda + b_-\right), \ \widetilde{\mu} = m_0\left(\lambda + b_+\right), \tag{25}$$

$$b_\mp = \frac{r_1}{4r_2} \mp \frac{\sqrt{r_1^2 + 4r_0 r_2}}{4r_2}, \tag{26}$$

$$m_0^2 = \frac{2r_2}{r_0 - \nu r_1 - \nu^2 r_2}, \tag{27}$$

$$\alpha_0 = \frac{r_0 - 2r_1\lambda - 4r_2\lambda^2}{2(\nu + 2\lambda)}, \quad \alpha_1 = \frac{r_0 + \nu r_1 + 2\nu r_2 \lambda}{\nu + 2\lambda}. \tag{28}$$

Spectral problem is generalization of the Wadati-Konno-Ichikawa (WKI) problem. Inverse transform technique for ($b_\pm = 0$) by K. Konno et al, (1981).

Reduction is : $\nu = r_1 = 0$, $Im(i\mathcal{U}) = 0 \Rightarrow$

$$\frac{\partial^2 \theta}{\partial \tau \partial \chi} = \sin\theta + r_2 \frac{\partial^2}{\partial \tau^2} \sin\theta, \tag{29}$$

$i\mathcal{U} = \partial_\tau \theta$. A. Fokas (1995). Eq. (29) transforms to Rabelo equations (R. Beals, M. Rabelo (1989)).

There are 4 sets of symmetries determines by constants.

I. Abnormal dispersion (subindex below 1):

$$r_2 < 0,\ m_0^2 < 0,\ r_1^2 - 4r_0|r_2| < 0. \tag{30}$$

$$b_{\mp} = \beta_0 \pm i\beta_1,\ \beta_1 = \frac{\sqrt{4r_0|r_2| - r_1^2}}{4|r_2|} \tag{31}$$

II. Normal dispersion (subindex 2):

$$r_2 > 0,\ m_0^2 > 0,\ r_1^2 + 4r_0 r_2 > 0. \tag{32}$$

$$b_{\mp} = \beta_0 \pm \beta_2,\ \beta_2 = \frac{\sqrt{4r_0 r_2 + r_1^2}}{4r_2}, \tag{33}$$

$$\beta_0 = \frac{r_1}{4r_2}. \tag{34}$$

$\lambda \longrightarrow \lambda - \beta_0$ + gauge transform:

$$\Phi_{1,2} = \begin{pmatrix} \mathrm{e}^{i\beta_0\tau} & 0 \\ 0 & \mathrm{e}^{-i\beta_0\tau} \end{pmatrix} \Phi_{1,2}, \tag{35}$$

$$\partial_\tau \Phi_1 = \begin{pmatrix} -i\lambda & (\lambda + i\beta_1)\, W_1 \\ -(\lambda - i\beta_1)\, W_1^* & i\lambda \end{pmatrix} \Phi_1, \tag{36}$$

where

$$W_1 = i|m_0| U \mathrm{e}^{-2i\beta_0\tau}, \tag{37}$$

and

$$\partial_\tau \Phi_2 = \begin{pmatrix} -i\lambda & (\lambda+\beta_2)\, W_2 \\ (\lambda-\beta_2)\, W_2^* & i\lambda \end{pmatrix} \Phi_2, \tag{38}$$

where

$$W_2 = |m_0| U \mathrm{e}^{-i2\beta_0\tau}. \tag{39}$$

Introduce the new variables T, Θ and the new functions $F_{1,2}(\chi,\tau)$, $G_{1,2}(\chi,\tau)$ as

$$T = \int_0^\tau G_1^{-1}\chi,\tau')d\tau', \ \Theta = \int_0^\tau G_2^{-1}(\chi,\tau')d\tau', \tag{40}$$

$$F_{1,2} = \frac{W_{1,2}}{\sqrt{1 \pm |W_{1,2}|^2}}, \ G_{1,2} = \frac{1}{\sqrt{1 \pm |W_{1,2}|^2}}. \tag{41}$$

Then

$$\partial_T \Phi_1 = \begin{pmatrix} -i\lambda G_1 & (\lambda+i\beta_1)F_1 \\ -(\lambda-i\beta_1)F_1^* & i\lambda G_1 \end{pmatrix} \Phi_1, \tag{42}$$

and

$$\partial_\Theta \Phi_2 = \begin{pmatrix} -i\lambda G_2 & (\lambda+\beta_2)F_2 \\ (\lambda-\beta_2)F_2^* & i\lambda G_2 \end{pmatrix} \Phi_1, \tag{43}$$

$$F_{1,2} = 0, \quad S_z = -1, \quad S = 0, \quad T,\Theta \to \pm\infty. \tag{44}$$

The ISTM applications by means of solution of the Marchenko equations or Riemann-Hilbert problem give solitons in implicit form.

Abnormal dispersion ($r_2 < 0$)

$$F_1(\tau,\chi) = \frac{2\zeta_0 \mathrm{e}^{-ic_1 - i\psi_1}\cosh(\psi)}{\cosh(\psi)^2 + \zeta_0^2}, \tag{45}$$

$$G_1(\tau,\chi) = \frac{\cosh(\psi)^2 - \zeta_0^2}{\cosh(\psi)^2 + \zeta_0^2}, \tag{46}$$

$$\psi = 2\eta\left(T - \frac{r_0' - 4|r_2|\eta^2}{\nu'^2 + 4\eta^2}\chi\right) + c_0, \tag{47}$$

$$\psi_1 = \nu'\frac{r_0' - 4|r_2|\eta^2}{\nu'^2 + 4\eta^2}\chi, \tag{48}$$

$$\zeta_0 = \frac{\eta}{\eta + |\beta_1|}. \tag{49}$$

Time dependence is obtained by integration of $\partial_\tau T = G_1^{-1}(\tau)$.

Soliton solution

$$\psi - \frac{2\zeta_0}{\sqrt{\zeta_0^2+1}}\mathrm{arctanh}\left[\frac{\zeta_0}{\sqrt{\zeta_0^2+1}}\tanh[\psi]\right] = 2\eta\left[\tau - \tau_0(\chi)\right], \tag{50}$$

$\tau_0(\chi) = -\psi(\tau = 0)/(2\eta)$.

$$W = \frac{F_1(\tau,\chi)}{G_1(\tau,\chi)} = \frac{-2i\zeta_0 \mathrm{e}^{2i\beta_0\tau - ic_1 - i\psi_1}\cosh(\psi)}{|m_0|\left[\cosh^2(\psi) - \zeta_0^2\right]}, \tag{51}$$

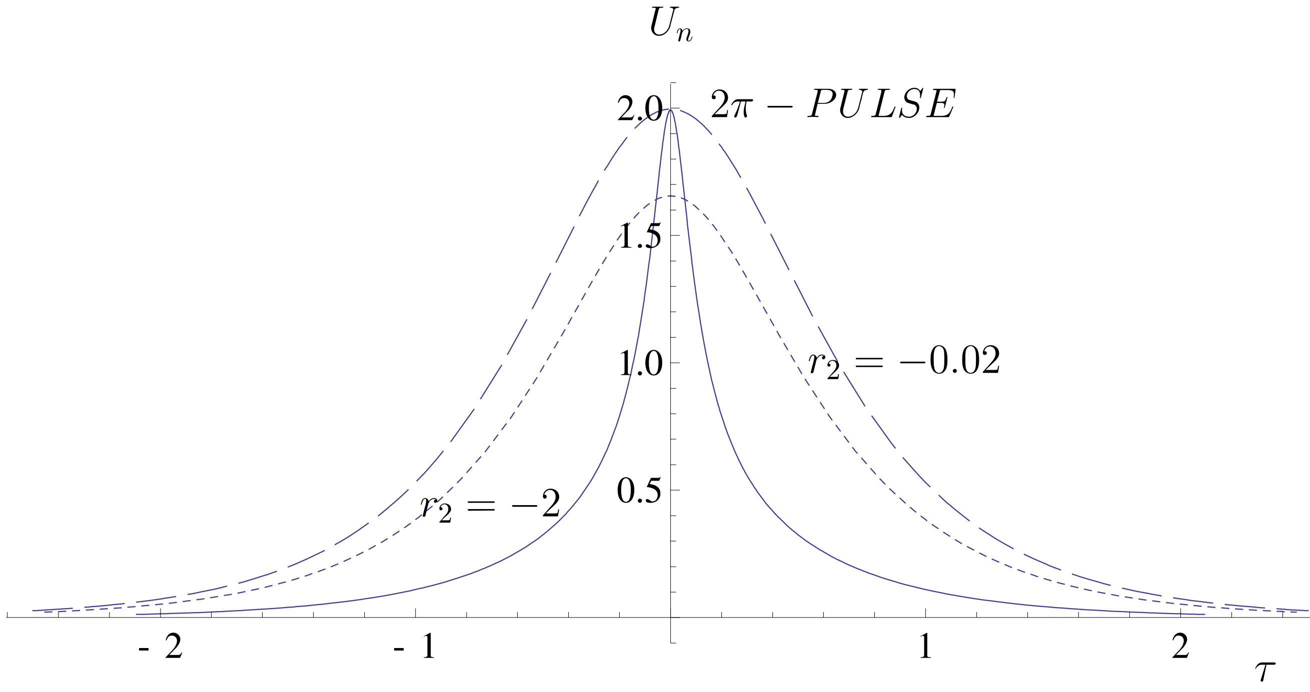

Figure 1: $\eta = 1.0$. Modulus of the soliton amplitude U_a vs τ is shown by the dashed line for $|r_2| = 0.0001$, by pointed line for $|r_2| = 0.02$, and by solid line for $|r_2| = 2.0$.

III. SOLITON. NORMAL DISPERSION ($r_2 > 0$)

$$C_2(\chi) = b_2(\chi; i\eta)/\partial_\lambda a_2(\chi; \lambda)|_{\lambda=i\eta} \neq 0, \tag{52}$$

where $a_2(\chi; i\eta) = 0$.

Denote

$$\frac{|C|^2}{\omega^2} \mathrm{e}^{-2i(\lambda^*-\lambda)(\Theta - V_1\chi)} = \mathrm{e}^{-2\theta}, \tag{53}$$

$$\frac{\beta_2 - \lambda^*}{\beta_2 + \lambda} = \mathrm{e}^{-2i\delta}, \tag{54}$$

$$\kappa = \frac{\mathrm{Im}\lambda}{|\beta_2 + \lambda|}, \tag{55}$$

$$C(0) = \mathrm{e}^{i\delta_1}|C(0)|. \tag{56}$$

Then the soliton is

$$D_2(\tau) = \frac{|\cosh(\theta + i\delta)|^2 + \kappa^2}{|\cosh(\theta + i\delta)|^2 - \kappa^2} \tag{57}$$

$$F_2(\tau) = \frac{2\kappa \cosh\left(\theta + i\delta\right) \mathrm{e}^{2iV_2\chi - i\delta - i\delta_1}}{|\cosh(\theta + i\delta)|^2 - \kappa^2}. \tag{58}$$

For $\mathrm{Im}\,\eta = 0, r_1 = 0, \nu = 0$ we have $V_2 = 0$, $\delta = 0$, $\beta_2 = 1/2\sqrt{r_0/r_2}$,

$$\kappa = \frac{2\eta\sqrt{r_2}}{\sqrt{4\eta^2 r_2 + r_0}}, \tag{59}$$

$$\theta = 2\eta\left[\Theta - \frac{r_0 + 4r_2\eta^2}{4\eta^2}\chi\right]. \tag{60}$$

The modulus of the soliton

$$U_n(\tau,\chi) = \frac{2\eta\sqrt{1+4\eta^2 r_2}\cosh(\theta)}{(1+4\eta^2 r_2)\cosh^2(\theta) + 4\eta^2 r_2}, \tag{61}$$

θ is found by integration of $D_2^{-1} = \partial_\tau \Theta$.

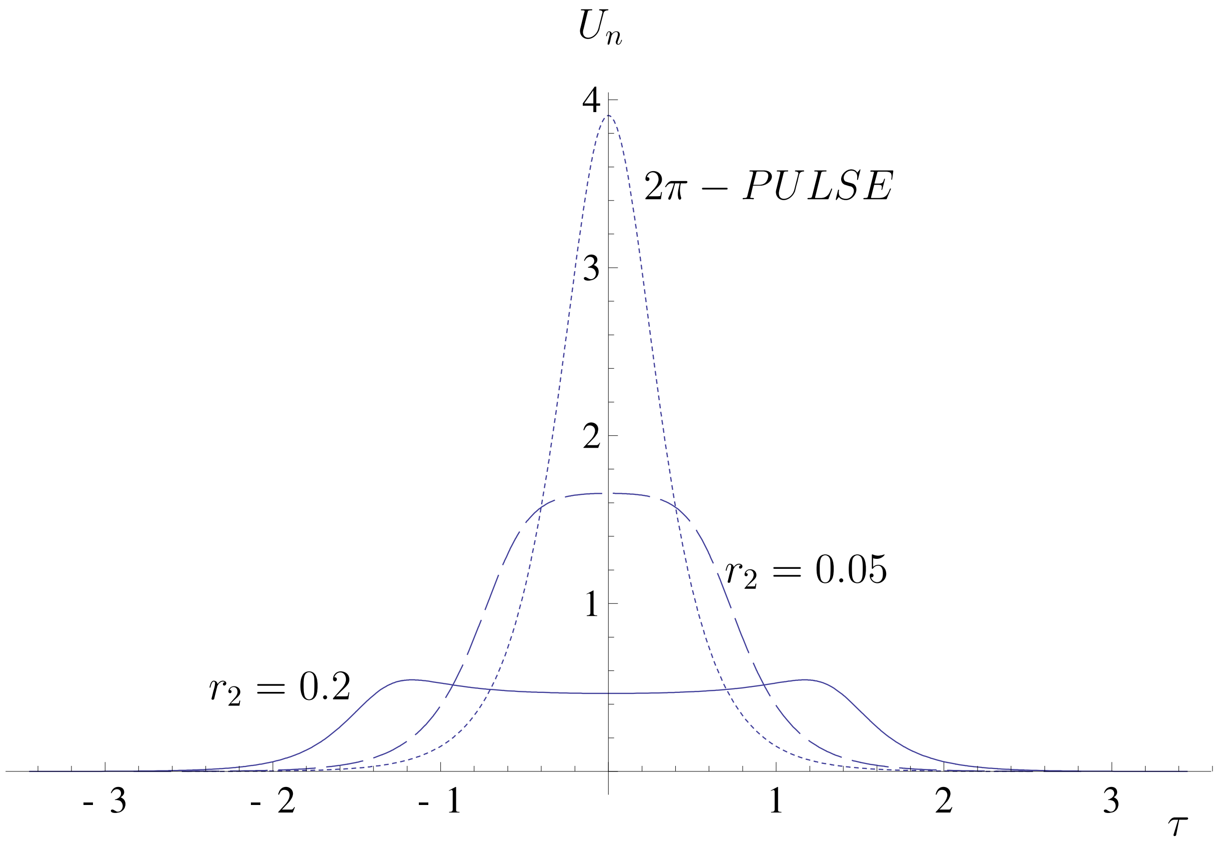

Figure 2: $\eta = 2.0$. Modulus of the soliton amplitude U_n vs τ is shown by the pointed line for $r_2 = 0.001$, by dashed line for $r_2 = 0.05$, and by solid line for $r_2 = 0.2$ (Topless soliton).

IV. A FAMILY OF INTEGRABLE MODELS OF DISPERSIVE MAXWELL-BLOCH EQUATIONS TYPE. AKNS-TYPE THEORY

Field is linearly polarized a few cycle pulses. The Maxwell-Bloch equations for the TLS with a permanent dipole momentum:

$$\frac{\partial R_1}{\partial \tau} = -\left(1 - \mu\mathcal{E}\right) R_2, \tag{62}$$

$$\frac{\partial R_2}{\partial \tau} = \left(1 - \mu\mathcal{E}\right) R_1 - \mathcal{E} R_3, \tag{63}$$

$$\frac{\partial R_3}{\partial \tau} = \mathcal{E} R_2, \tag{64}$$

$$\frac{\partial \mathcal{E}}{\partial \chi} = R_2 + \frac{\gamma^2}{4}\frac{\partial^2}{\partial \tau^2} R_2, \tag{65}$$

where $\tau = \omega(t - x/c)$, $\mu = (d_{11} - d_{22})/(2d_{12})$, d_{ij} is matrix dipole momentum. $\vec{R}$ is the Bloch vector:

$$\mathcal{E} = \frac{2d_{12}U}{\hbar\omega}, \tag{66}$$

$$R_1 = \rho_{12} + \rho_{21}, \tag{67}$$

$$R_2 = -i(\rho_{12} - \rho_{21}), \tag{68}$$

$$R_3 = \rho_{11} - \rho_{22}. \tag{69}$$

$$\chi = \frac{4\pi d_{12}^2 N}{c\hbar} x. \tag{70}$$

The zero-curvature representation is

$$\partial_\tau \Phi = \begin{pmatrix} -i\lambda & i\left(1-\gamma\lambda\right)F \\ i\left(1+\gamma\lambda\right)F & i\lambda \end{pmatrix} \Phi, \tag{71}$$

$$\partial_\chi \Phi = \frac{1+\mu^2}{1-4(1+\mu^2)\lambda^2} \begin{pmatrix} a & a_- \\ a_+ & -a \end{pmatrix} \Phi, \tag{72}$$

where

$$\begin{aligned} &a = i\lambda\left(1-\gamma^2\lambda^2\right)\left(\mu R_1 + R_3\right), \\ &a_\mp = \frac{1\mp\gamma\lambda}{\sqrt{1+\mu^2}}\left[\frac{i(1-\lambda^2\gamma^2)}{\sqrt{4(1+\mu^2)-\gamma^2}}\left(\mu R_3 - R_1\right)\right. \\ &\left.\mp\frac{1}{2}\sqrt{4(1+\mu^2)-\gamma^2}R_2\right], \end{aligned}$$

$$F = \frac{\mathcal{E}\sqrt{1+\mu^2}}{\sqrt{4(1+\mu^2)-\gamma^2}} - m, \tag{73}$$

$m = \mu\{(1+\mu^2)[4(1+\mu^2)-\gamma^2]\}^{-1/2}$.

$\Phi(\tau,\chi,\lambda)$, λ are the matrix valued function and the spectral parameter, respectively.

V. SECOND HARMONIC GENERATION IN DISPERSIVE MEDIUM

The equations are:

$$\frac{\partial V}{\partial \tau} = i\nu V + iUV^*, \tag{74}$$

$$\frac{\partial U}{\partial \chi} = ir_0 V^2 + r_1 \frac{\partial V^2}{\partial \tau} + ir_2 \frac{\partial^2 V^2}{\partial \tau^2}, \tag{75}$$

$\nu, r_k, k = 1, 2, 3 \in \mathbb{R}$.

Introduce $S = V^2, S_z = |V^2|$ and rewrite as

$$\frac{\partial S}{\partial \tau} = i\nu S + 2iUS_z, \tag{76}$$

$$\frac{\partial S_z}{\partial \tau} = i\left(US^* - U^* S\right), \tag{77}$$

$$\frac{\partial U}{\partial \chi} = ir_0 S + r_1 \frac{\partial S}{\partial \tau} + ir_2 \frac{\partial^2 S}{\partial \tau^2}, \tag{78}$$

here $\tau = (t - x/c)\omega_R,\ \nu = (\omega_0 - \omega_{12})/\omega_R,\ S_z = \rho_{11} - \rho_{22}$, $r_0 = q_0,\ r_1 = -q_1\omega_R,\ r_2 = -q_2\omega_R^2\ \mathcal{U} = d_{12}\mathcal{E}/(\hbar\omega_R)$

$$r_0 = q_0,\ r_1 = -q_1\omega_R,\ r_2 = -q_2\omega_R^2, \tag{79}$$

$$\chi = \frac{2\pi d_{12}^2 \omega_0 N}{c\hbar\omega_R} x. \tag{80}$$

For $r_2 \neq 0, r_0 + \nu r_1 - \nu^2 r_2 \neq 0$ the zero-curvature representation for (76) – (78) is

$$\partial_\tau \Phi = \begin{pmatrix} -i\lambda & m_0 (\lambda + b_-) U \\ (\lambda + b_+) U^* & i\lambda \end{pmatrix} \Phi, \tag{81}$$

$$\mu = m_0 (\lambda + b_-), \quad \widetilde{\mu} = m_0 (\lambda + b_+), \tag{82}$$

$$b_\mp = \frac{r_1}{4r_2} \mp \frac{\sqrt{r_1^2 + 4r_0 r_2}}{4r_2}, \tag{83}$$

$$m_0^2 = \frac{-4r_2}{r_0 + \nu r_1 - \nu^2 r_2}, \tag{84}$$

and

$$\partial_\chi \Phi = \mathbf{A}\Phi \equiv \begin{pmatrix} i a_0 S_z & \mu (a_1 S + i r_2 \partial_\tau S) \\ \widetilde{\mu} (a_1 S^* - i r_2 \partial_\tau S^*) & -i a_0 S_z \end{pmatrix} \Phi, \tag{85}$$

$$a_0 = \frac{-r_0 + 2r_1\lambda + 4r_2\lambda^2}{\nu + 2\lambda}, \tag{86}$$

$$a_1 = \frac{2(r_0 + \nu r_1 + 2\nu r_2 \lambda)}{\nu + 2\lambda}. \tag{87}$$

Where $\Phi(\tau, \chi, \lambda)$ is the 2×2 matrix-valued function and λ is a spectral parameter.

Thank you for attention!