Resonant control of envelope solitons

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Subject of the talk

- We will consider a soliton of the nonlinear Schrödinger (NLS) equation.
- Our goal: to control parameters of the soliton.
- The following two approaches will be discussed:
    1. Autoresonance
    2. Scattering on resonance

- We consider a nonlinear Schrödinger equation (NLS):

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = \varepsilon f(x,t) \qquad (\varepsilon \ll 1)$$

which has a solitary wave solution:

$$u(x,t) = \frac{A}{\operatorname{ch} A(x-\xi)}\, e^{i\Phi}, \quad \Phi = V\,(x-\xi) + \theta.$$

- For unperturbed soliton ($\varepsilon = 0$):

$$\xi = V\,t + \xi_0, \quad \theta = \omega\,t + \theta_0, \quad \omega = \frac{A^2+V^2}{2}.$$

- We should find perturbations εf suitable for control of the amplitude A and/or velocity V of the soliton.

- The perturbation $\varepsilon f(x,t)$ is not supposed to be localized, so there is a non-localized background part $\chi(x,t) \sim \varepsilon$ of the solution:

$$u(x,t) = \varphi(x,t) + \chi(x,t),$$

which approximately obeys a linear equation

$$i\chi_t + \frac{1}{2}\chi_{xx} = \varepsilon f(x,t),$$

while the localized part $\varphi(x,t)$ satisfies

$$i\varphi_t + \frac{1}{2}\varphi_{xx} + |\varphi|^2\varphi = -\chi^*\varphi^2 - 2\chi|\varphi|^2.$$

- Unlike the original NLS equation, the perturbation is localized on soliton here, so it never becomes larger than the perturbed wave itself.
(This procedure was suggested by E.M. Maslov, IZMIRAN, Moscow).

- Variational principle:

$$\delta \int \underline{\int \mathcal{L}\, dx}\, dt = 0,$$

$$\mathcal{L} = \frac{1}{2}\left[i(\varphi\,\varphi_t^* - \varphi^*\,\varphi_t) + |\varphi_x|^2 - |\varphi|^4\right] - |\varphi|^2(\varphi\chi^* + \varphi^*\chi).$$

- Adiabatic approximation: perturbation causes only a slow evolution of soliton parameters.
- Reduced variational principle

$$\delta \int L\, dt = 0,$$

$$L = A\left(2\theta_t - 2V\xi_t + V^2 - \frac{1}{3}A^2\right) - A^3\left(I\, e^{i\theta} + I^* e^{-i\theta}\right),$$

where

$$I(A, V, \xi, t) = \int_{-\infty}^{\infty} \frac{e^{i\,V\,s}}{\mathrm{ch}^3\,\mathrm{A\,s}}\, \chi^*(s + \xi, t)\, ds.$$

One phase perturbation

- Consider the following simple perturbation:

$$\varepsilon f = \varepsilon e^{i\,\psi(t)},$$

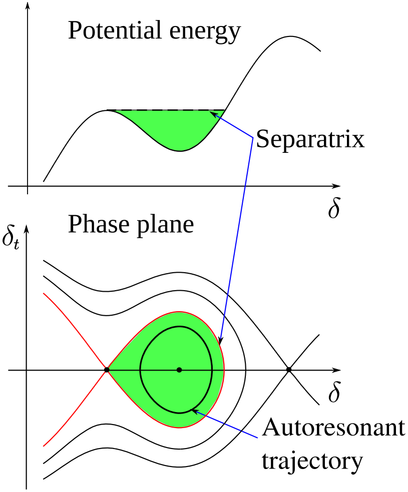
$$\psi(t) = \int \Omega(t)dt, \quad \Omega(t) = \Omega_0 + \alpha\, t.$$

- Frequency Ω may change in time.
- Variational equations (assuming $V = 0$ for now):

$$A_t = -\varepsilon \frac{\pi A^2}{2\Omega} \sin\delta,$$
$$\delta_t = \frac{A^2}{2} - \Omega - \varepsilon \frac{\pi A}{\Omega} \cos\delta,$$

where $\delta = \theta - \psi$ is the soliton-perturbation phase mismatch.

Autoresonance – locking of phase oscillations



- Nonlinear pendulum:

$$\delta_{tt} + \varepsilon\pi A \sin\delta + \alpha \approx 0.$$

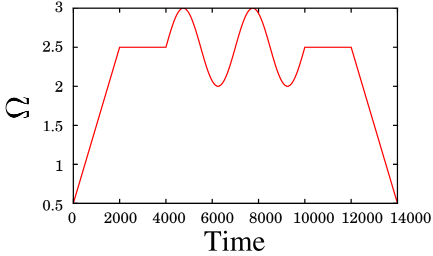
- $A \approx$ const on one period of phase oscillations $\Delta t \sim 1/\sqrt{\varepsilon}$.
- For autoresonant trajectory:

$$|\delta_t| < const \Rightarrow \frac{A^2}{2} \approx \Omega(t).$$

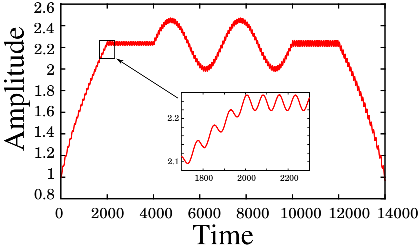
- therefore in autoresonance the soliton amplitude is determined by the frequency of perturbation.

Autoresonance of NLS soliton: numerical simulation

Input: perturbation frequency



Output: soliton amplitude



What we basically need to do to control the soliton:

- Start in (or "close to") resonance: $\Omega(0) \approx \frac{A^2(0)}{2}$;
- Start in phase: $\psi(0) \approx \theta(0) - \delta^*$, where δ^* – stationary point;
- Change the frequency slowly enough: $\left|\frac{d\Omega}{dt}\right| < \varepsilon \pi A$.

Two phase perturbation

- To control both amplitude and velocity of the soliton one need two phase perturbation:

$$\varepsilon f(x,t) = \varepsilon e^{i\psi(t)}\left(1 + g\, e^{ik(x-X(t))}\right).$$

- Functions $\Omega(t) = \psi_t$ and $U(t) = X_t$ are slow:

$$\Omega_t = \beta_1 \varepsilon, \quad U_t = \beta_2 \varepsilon.$$

- Lagrange function:

$$L = 2A\left(\delta_t + \Omega - \frac{V}{k}\phi_t - V\,U + V^2 - \frac{A^2}{6}\right) + \\ + \frac{\varepsilon A^2}{\Omega}\left\{F\left(\frac{V}{A}\right)\cos\delta + G\,F\left(\frac{V-k}{A}\right)\cos(\delta-\phi)\right\},$$

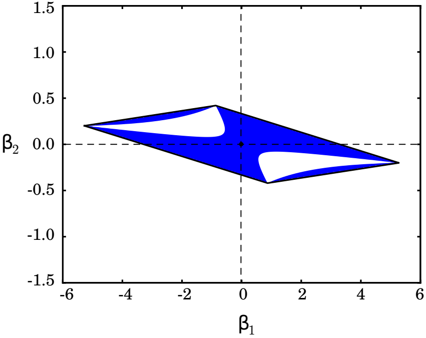
where new phases $\delta = \theta - \psi$, $\phi = k(\xi - X)$, $G \propto g$.

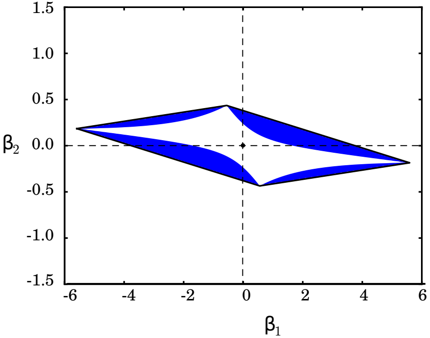
- There are four variational equations for A, V, δ and ϕ.
- The dynamics of phases can be aproximately described by a two coupled nonlinear pendulums:

$$\delta_{\tau\tau} = a \sin\delta + b \sin(\delta - \phi) - \beta_1,$$
$$\phi_{\tau\tau} = c \sin\delta + d \sin(\delta - \phi) - k\beta_2,$$

  where a, b, c, d depend on soliton and perturbation parameters, $\tau = \sqrt{\varepsilon}\, t$ – slow time.
- Autoresonance (phase locking) is only possible if
  - there are stationary points (δ^*, ϕ^*) such that $\delta_{\tau\tau} = \phi_{\tau\tau} = 0$;
  - they are stable.

Regions of phase locking





- Stationary points exist only inside the rhombus on the (β_1, β_2) plane.
- They are stable inside of blue areas.
- For some parameters the soliton cannot be phase locked by the steady drive with $\beta_1 = \beta_2 = 0$ (right figure).

To control both amplitude and velocity of soliton one need to

- Start in resonance between soliton and perturbation:

$$\Omega(0) = \frac{A^2(0) + V^2(0)}{2}, \quad U(0) = V(0).$$

- Start in phase:

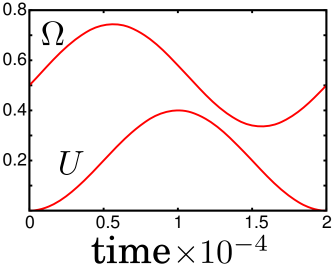
$$\psi(0) = \theta(0) + \delta^*, \quad X(0) = \xi(0) + \frac{\phi^*}{k},$$

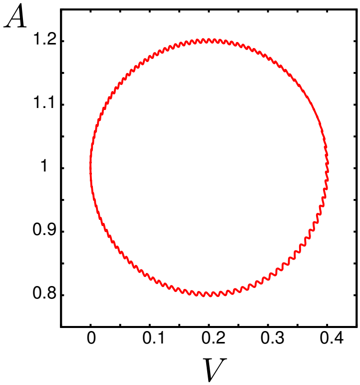
  where (δ^*, ϕ^*) is a stationary point.
- Frequencies should change “slowly enough”.

Numerical simulation

Input:
perturbation frequencies

Output:
soliton parameters





What if autoresonance could not be used?

- To be able to use autoresonance we should know
  - Amplitude, A
  - Velocity, V
  - Phase, θ
  - Coordinate, ξ

  of soliton.
- What if we do not know these parameters?
- Scattering on resonance is an alternative approach.

Soliton amplification by multiple scattering

- Consider, for simplicity, standing solitons: $V = 0$.
- If autoresonance conditions are not met, a single pass through resonance causes a change of soliton amplitude

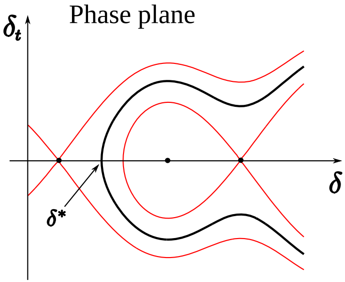
$$\Delta A \propto \sqrt{\varepsilon}.$$

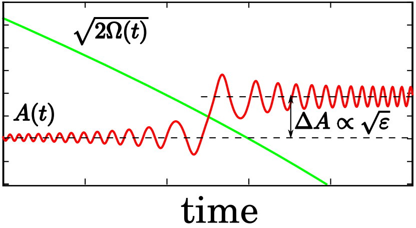
  This phenomenon is called scattering on resonance (I.M. Lifshitz, V.I. Arnold, A.I. Neishtadt)
- Multiple passes through resonance should allow us to increase soliton amplitude significantly if

$$\Delta A > 0$$

  independently of its phase.

Single scattering on resonance





- Nonlinear pendulum approximation:

$$\delta_{\tau\tau} + \pi A(0) \sin\delta + \alpha/\varepsilon = 0,$$
$$\Omega = \Omega_0 + \alpha\, t.$$

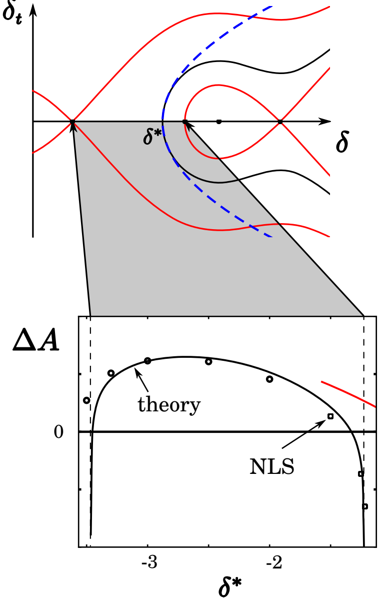
- Increment of amplitude:

$$\Delta A(\delta^*, \lambda) = -\pi\sqrt{\varepsilon}\int_{-\infty}^{\infty} \sin\delta(\tau) d\tau$$

depends on the resonant phase δ^* and parameter

$$\lambda = \frac{\alpha}{\varepsilon\pi A(0)}.$$

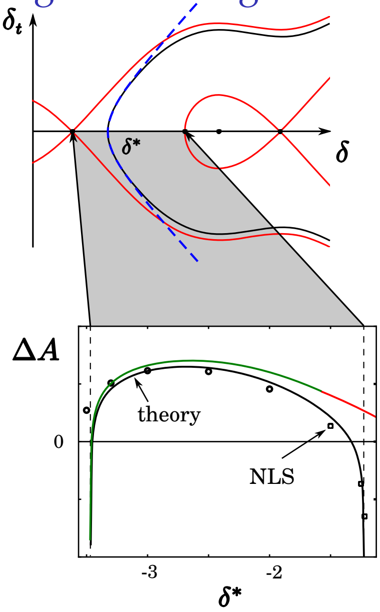
Single scattering on resonance



I. For $\delta^* > -\frac{\pi}{2}$ we use parabolic approximation:

$$(\Delta A)_0 = -\sqrt{\varepsilon\pi^3}\,\frac{\operatorname{sign}\alpha\ \cos\delta^* + \sin\delta^*}{|\pi A(0)\sin\delta^* - \alpha/\varepsilon|}$$

Single scattering on resonance



II. Near the left saddle point (for $\delta^* < -\frac{\pi}{2}$) we use hyperbolic approximation and obtain:

$$(\Delta A)_1 = \frac{\sqrt{\varepsilon}\pi^2}{\sqrt{B}} \left[\sin\bar{\delta} \cdot \mathrm{N}_0\left(\frac{|C|}{B}\right) - \right.$$

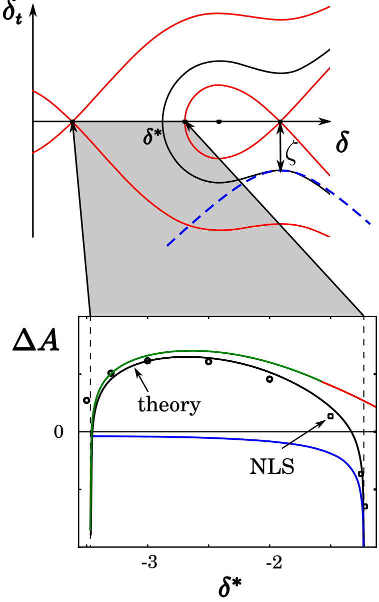
$$\left. -\operatorname{sign}\alpha \cdot \cos\bar{\delta} \cdot \mathrm{J}_0\left(\frac{|C|}{B}\right)\right],$$

$$\bar{\delta} = \delta^* + \frac{C}{B},$$

$$C = \pi A(0)\sin\delta^* - \frac{\alpha}{\varepsilon},$$

$$B = -\pi A(0)\cos\delta^*.$$

Single scattering on resonance



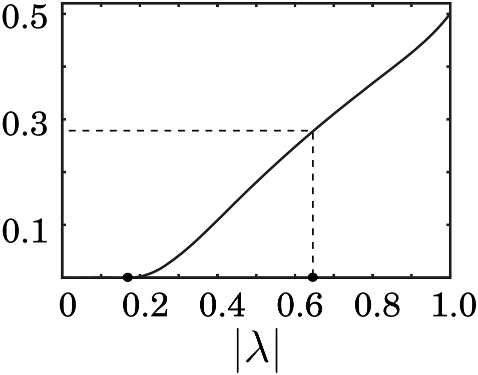
III. Hyperbolic approximation to take into account right saddle point:

$$(\Delta A)_2 = \sqrt{\varepsilon}\frac{2\pi\lambda}{\kappa}\mathrm{K}_0\left(\frac{\zeta}{\kappa}\right)$$

where

$$\kappa^2 = \pi A(0)\sqrt{1-\lambda^2}$$

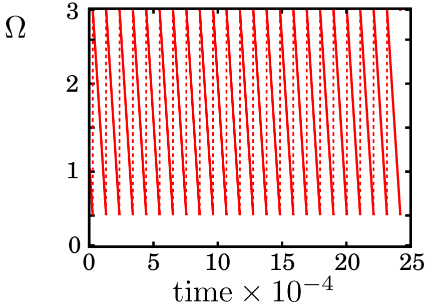
Fraction of resonantly scattered phase trajectories that correspond to decrement of soliton amplitude $\Delta A < 0$



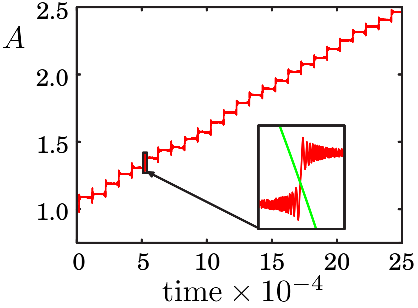
- If $|\lambda| \lesssim 0.2$, then $\Delta A > 0$ for almost any resonant phase δ^*.
- If $|\lambda| \gtrsim 1$, then $\Delta A < 0$ for nearly half of the resonant phases δ^*.

NLS soliton amplification with $|\lambda| = 0.16$

Input:
perturbation frequency

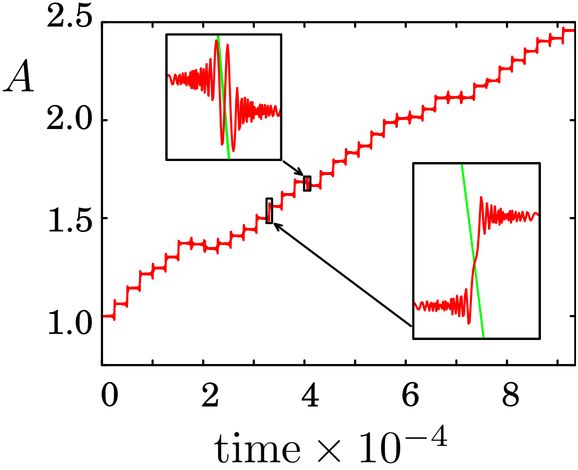


Output:
soliton amplitude



- Amplification process is regular when $|\lambda| \lesssim 0.2$.

NLS soliton amplification with $|\lambda| = 0.64$



- Both increasing and decreasing of soliton amplitude is observed.

Summary

We considered two ways to control NLS soliton:

- Autoresonance allows to control both amplitude and velocity of soliton using one- or two-phase perturbation.
- Resonant scattering can also be used for soliton amplification (and, probably, acceleration).

Both methods complement each other:

- Autoresonance allows for a fine control, but requires much knowledge of initial state of the soliton.
- Resonant scattering is a bit rough, but does not assumes exactly known initial state of the soliton.

Both phenomena has a very common nature and may be used to control other nonlinear waves.