Finite Element Modeling of the Surface Water Waves

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Mathematical statement

Consider a flow of ideal incompressible fluid in 2-D domain Ω:

$$-b \le y \le \eta(x,t), \quad 0 \le x \le a, \quad t \ge 0. \tag{1}$$

Potential motion is assumed:

$$v(x,y,t) = \nabla\phi(x,y,t). \tag{2}$$

From fluid incompressibility condition div $v = 0$ it follows

$$\Delta\phi(x,y,t) = 0. \tag{3}$$

Boundary and initial conditions:

$$\begin{gathered} (\eta_t + \phi_x \eta_x - \phi_y)\big|_{y=\eta(x,t)} = 0, \\ (\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + gy)\big|_{y=\eta(x,t)} = 0, \\ \tfrac{\partial\phi}{\partial n}\big|_{y=-b} = 0, \quad \phi(x,y) = \phi(x+a,y), \\ \eta\big|_{t=0} = \eta_0(x), \quad \phi\big|_{t=0} = \phi_0(x,y). \end{gathered} \tag{4}$$

Hamiltonian system formulation

Hamiltonian H is equal to the full energy of the fluid:

$$H(\eta,\phi) = \frac{1}{2}\int\limits_0^a dx \int\limits_{-b}^{\eta(x,t)} |\nabla\phi|^2 dy + \frac{g}{2}\int\limits_0^a \eta^2(x,t)dx. \tag{5}$$

Let $\psi(x,t) = \phi(x,\eta(x,t),t)$, then

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \tag{6}$$

where functional derivative

$$\frac{\delta F(\varphi(x))}{\delta\varphi(y)} = \lim_{\substack{|\Delta r|\to 0\\ \sup|\delta\varphi|\to 0}} \frac{F(\varphi(x)+\delta\varphi(x-y)) - F(\varphi(x))}{\int_{\Delta r}\delta\varphi(x-y)dx} =$$

$$= \frac{\partial\left(F(\varphi(x)+\varepsilon\delta(x-y)\right)}{\partial\varepsilon}\bigg|_{\varepsilon=0} \tag{7}$$

Volume and energy conservation laws:

$$\int\limits_0^a \eta(x,t)dx = 0, \quad \frac{dH}{dt} = 0. \tag{8}$$

Structured dynamic meshes

At any moment consider splitting of the domain into $M \times N$ quadrilaterals ($V = (N+1)(M+1)$ vertices):

$$\begin{aligned} x_i &= a\frac{i}{M}, & i &= 0, \ldots, M, \\ y_j &= (\eta(x_i, t) + b)\frac{j}{N}, & j &= 0, \ldots, N. \end{aligned} \tag{9}$$

Top boundary is approximated by a polyline with vertices at points $(x_i, \eta(x_i, t))$

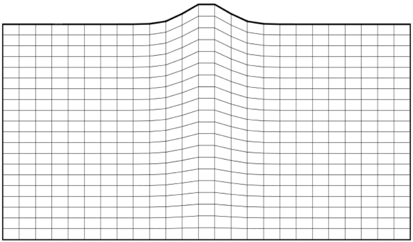


Figure 1: Sample mesh in computational domain

Finite element approximations

- Each quadrilateral is divided into 2 triangles
- First order nodal basis functions on triangular elements are introduced: $\xi_{i,j}(x,y)$, equal to 1 at (x_i, y_j), $\mathrm{span}\{\xi_{i,j}\} \subset H^1$.
- At each time step k

$$\phi^k(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} u_{i,j}^k \xi_{i,j}^k(x,y). \tag{10}$$

- Finite element variational problem for ϕ^k:

$$\begin{gathered}\sum_{i=0}^{M} \sum_{j=0}^{N} u_{i,j}^k \int\limits_{\Omega} (\nabla \xi_{i,j} \cdot \nabla \xi_{m,n}) \, d\Omega = 0, \; m \in [0; M), \; n \in [0; N), \\ \begin{aligned} u_{m,N}^k &= \psi_m^k, \quad m = 0, \dots, M, \\ u_{M,n}^k &= u_{0,n}^k, \quad n = 0, \dots, N. \end{aligned}\end{gathered} \tag{11}$$

Finite element Hamiltonian

$$H = H(\eta_0, \eta_1, \dots, \eta_M, \psi_0, \psi_1, \dots, \psi_M). \tag{12}$$

Time integration scheme

Consider finite difference *energy conserving* scheme

$$\frac{\eta_i^{k+1}-\eta_i^k}{\tau} = \frac{M}{a}\cdot\frac{H(\bar{\eta}_i^k,\bar{\psi}_i^{k+1})-H(\bar{\eta}_i^k,\bar{\psi}_i^k)}{\psi_i^{k+1}-\psi_i^k}, \quad (13)$$

$$\frac{\psi_i^{k+1}-\psi_i^k}{\tau} = \frac{M}{a}\cdot\frac{H(\bar{\eta}_i^k,\bar{\psi}_i^{k+1})-H(\bar{\eta}_i^{k+1},\bar{\psi}_i^{k+1})}{\eta_i^{k+1}-\eta_i^k}, \quad (14)$$

with $\bar{\eta}_i^k=[\eta_0^{k+1},\dots,\eta_{i-1}^{k+1},\eta_i^k,\dots,\eta_M^k]$, $\bar{\eta}_i^{k+1}=[\eta_0^{k+1},\dots,\eta_i^{k+1},\eta_{i+1}^k,\dots,\eta_M^k]$.
Let $G_i(x,y)$ be FEM solution of $\Delta G_i=0$ with boundary conditions $\psi_m=\delta_{m,i}$ at the top. Then

$$\varphi(x,y)=\sum_{i=0}^{M}\psi_i G_i(x,y), \quad \nabla\varphi(x,y)=\sum_{i=0}^{M}\psi_i\nabla G_i(x,y), \quad (15)$$

$$\frac{\eta_i^{k+1}-\eta_i^k}{\tau}=\frac{M}{a}\left(\frac{\psi_i^{k+1}-\psi_i^k}{2}\int_\Omega\nabla G_i\cdot\nabla G_i d\Omega+\int_\Omega\nabla\bar{\varphi}^k\cdot\nabla G_i d\Omega\right), \quad (16)$$

with $\bar{\varphi}^k$ being a solution with $\bar{\psi}_i^k$ as boundary conditions.
Scheme is non-linear — iterations over non-linearity required.

Numerical SLAE solution

FEM approximation of variational problem for ϕ^k results (after removing $u^k_{m,N}$ and $u^k_{M,n}$ variables) in sparse symmetric positive definite system of linear algebraic equations (SLAE)

$$Ax = b. \tag{17}$$

Need to solve multiple systems at each time step at each point $i = 0, \ldots, M-1$ due to non-linearity.

- Using Preconditioned Conjugate Gradient iterative solver
- Using Intel® MKL PARDISO sparse LL^T decomposition as preconditioner
  - LL^T decomposition is recomputed at each new i
  - Helps PCG converging in $\approx 3 \div 5$ iterations
- OpenMP parallelism is enabled in matrix-vector operations, integration over domain and PARDISO

Model problems

Consider $\psi_0(x) = 0$ and

$$\eta_0(x) = \frac{b}{4} \cdot \exp\left(-200\left(x - \frac{a}{2}\right)^2\right) - c, \tag{18}$$

$$c = \frac{b}{4} \int\limits_0^a \exp\left(-200\left(x - \frac{a}{2}\right)^2\right) dx. \tag{19}$$

Parameters: $b = 0.2$ m, $a = 2$ m, $g = 9.780327$ m/s^2.

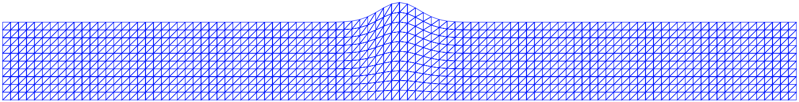
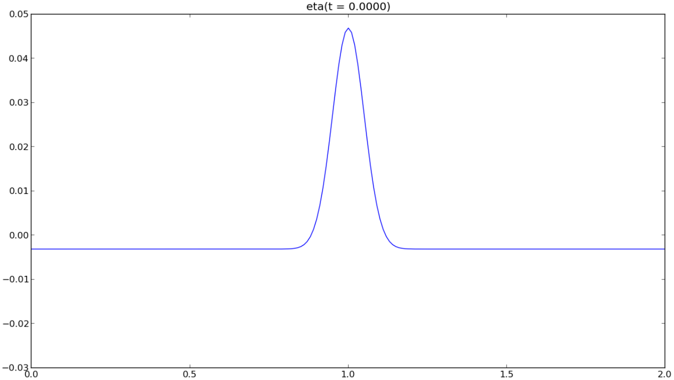


Figure 2: Mesh 100×10 at $t = 0$

Long simulation run

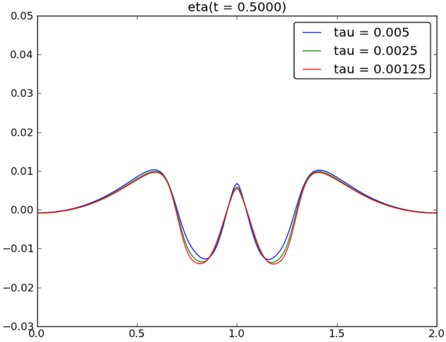
Mesh $= 200 \times 20$, time step $\tau = 0.0025$ s, simulation time $T = 5$ s. Relative volume change $\frac{\Delta V}{V_0} = 1.9 \cdot 10^{-2}$, energy $\frac{\Delta H}{H_0} = -5.8 \cdot 10^{-5}$.



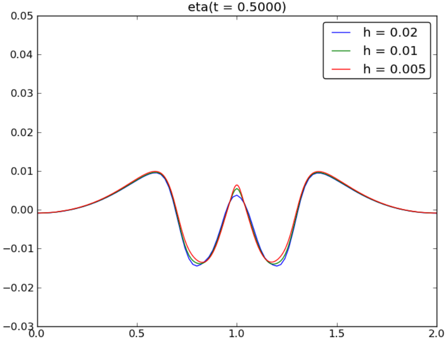
Convergence of the solutions

Analysis of $\eta(x, t)$ convergence in $L^2([0; a])$ norm:

- The scheme shows $O(\tau + h)$ convergence of the solutions



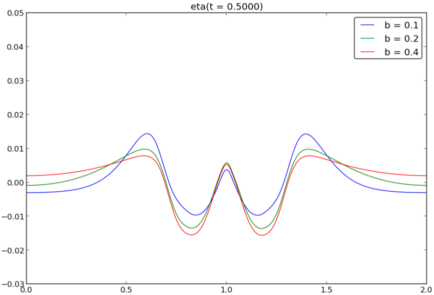
Mesh $= 200 \times 20$

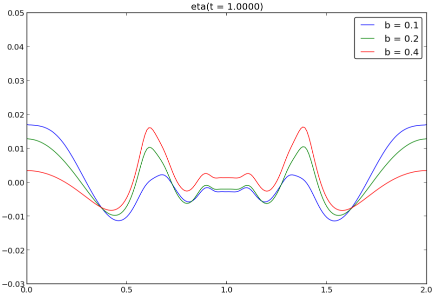


$\tau = 0.00125$

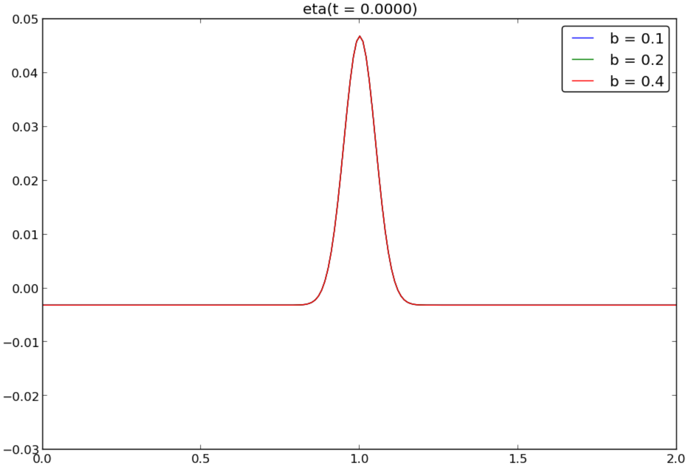
Simulations for various depths

Mesh $= 200 \times Y$, time step $\tau = 0.0025$. Different depths $b = 0.1, 0.2, 0.4$. Mesh splittings in y-direction are $Y = 10, 20, 40$.





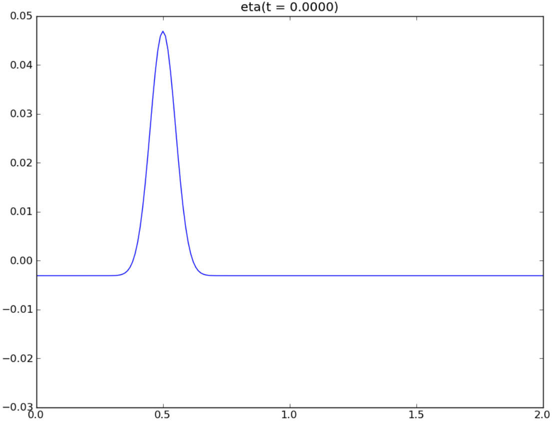
Simulations for various depths



Tank simulation

Mesh $= 200 \times 20$, time step $\tau = 0.0025$.

Wall boundary conditions at $x = 0$ and $x = a$: $\dfrac{\partial \varphi}{\partial \vec{n}} = 0$



Conclusion

- Energy-conserving scheme is presented allowing long simulation runs
- Finite element Hamiltonian differentiation is proposed
- Finite depth and non-periodic domains (tanks) can be simulated
- Scheme is computationally expensive

Future research:

- Further algorithm parallelization (including MPI)
- Dynamic meshes for non-linear domain bottom
- Several discrete conservation laws
- Wave breaking
- Higher order schemes

Thank you for your attention!

Q & A

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