About linear superpositions of special exact solutions of Veselov-Novikov (VN) equation

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ABSTRACT

In present talk new results obtained recently with my post graduate student A.V. Topovsky are reviewed:

• New exact solutions, nonstationary and stationary, of Veselov-Novikov (VN) equation in the forms of linear superpositions of arbitrary number of exact special solutions $u^{(n)}$, $n = 1, \ldots, N$ are constructed via $\overline{\partial}$-dressing method in such a way that the sums $u = u^{(k_1)} + \ldots + u^{(k_m)}$, $1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant N$ of arbitrary subsets of these solutions are also exact solutions of VN equation.

• The presented linear superpositions include as superpositions of special line solitons with zero asymptotic values at infinity and also superpositions of special plane wave type singular periodic solutions.

• By construction these exact solutions represent also new exact transparent potentials of 2D stationary Schrödinger equation and can serve as model potentials for electrons in planar structures of modern electronics.

INTRODUCTION I

Among different (2+1)-dimensional integrable nonlinear equations [1, 2, 3, 4, 5, 6, 7, 8, 9] prominent place takes the famous Veselov-Novikov (VN) equation [10, 11]:

$$u_t + \kappa u_{zzz} + \overline{\kappa} u_{\bar{z}\bar{z}\bar{z}} + 3\kappa(u\partial_{\bar{z}}^{-1}u_z)_z + 3\bar{\kappa}(u\partial_z^{-1}u_{\bar{z}})_{\bar{z}} = 0 \qquad (1.1)$$

where $u(z, \bar{z}, t)$ is scalar function, κ is some complex constant; $z = x + iy$, $\bar{z} = x - iy$; ∂_z^{-1} and $\partial_{\bar{z}}^{-1}$ are operators inverse to ∂_z and $\partial_{\bar{z}}$, $\partial_{\bar{z}}^{-1}\partial_{\bar{z}} = \partial_z^{-1}\partial_z = 1$.

VN equation can be represented as compatibility condition in the form of Manakov's triad [12]:

$$[L_1, L_2] = BL_1, \quad B = 3\left(\kappa\partial_{\bar{z}}^{-1}u_{zz} + \overline{\kappa}\partial_z^{-1}u_{\bar{z}\bar{z}}\right) \qquad (1.2)$$

Introduction II

of two linear auxiliary problems

$$L_1\psi = \left(\partial^2_{z\bar{z}} + u\right)\psi = 0, \qquad (1.3)$$

$$L_2\psi = \left(\partial_t + \kappa\partial_z^3 + \overline{\kappa}\partial_{\bar{z}}^3 + 3\kappa\left(\partial_{\bar{z}}^{-1}u_z\right)\partial_z + 3\overline{\kappa}\left(\partial_z^{-1}u_{\bar{z}}\right)\partial_{\bar{z}}\right)\psi = 0. \qquad (1.4)$$

Several classes of exact solutions of VN equation (1.1) have been constructed in last three decades (1980 – 2012) via different methods [10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], see also the books [3, 4].
These solutions include finite-zone type solutions (Veselov, Novikov, 1988) [
rationally localized solutions(Grinevich, Manakov,1986), (Grinevich, Noviko
1988) [13, 14, 15, 17, 18, 20, 21] or lumps.
There have been constructed also examples of solutions with functional parameters (Matveev, Sal, 1991), (Dubrovsky, Topovsky, Basalaev, 2011) [10, 16], multiple pole lump solutions(Dubrovsky, Formusatik, 2001,2003) [2
21] and so on.

Introduction III

Underline that the first auxiliary linear problem (1.3) is nothing but the 2D stationary Schrödinger equation so exact solutions of VN equation constructed via all IST approaches are also transparent potentials of this Schrödinger equation.
In present paper previously obtained result (Dubrovsky, Topovsky, Basalaev 2011) [24] (see also [22]) about linear superposition $u = u^{(1)} + u^{(2)}$ of two special solitons with zero asymptotic values $-\epsilon = 0$ at infinity or plane-wave periodic solutions $u^{(1)}$ and $u^{(2)}$ was generalized to the case of linear superpositions of arbitrary number of special line solitons (or special plane wave type periodic solutions) $u^{(n)}$, $n = 1, \ldots, N$ in such a way, that the sums of arbitrary subsets of these solutions

$$u = u^{(k_1)} + \ldots + u^{(k_m)}, \quad 1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant N \tag{1.5}$$

are also exact solutions of VN equation (1.1).

INTRODUCTION IV

For convenience here some useful formulas of $\overline{\partial}$-dressing method for VN equation (1.1) [4, 20, 21, 22, 23, 24] are presented. Central object of this method is the scalar wave function

$$\chi(\lambda; z, \bar{z}, t) = e^{-F(\lambda;z,\bar{z},t)}\psi(z, \bar{z}, t),\ F(\lambda; z, \bar{z}, t) = i\left[\lambda z - \frac{\epsilon}{\lambda}\bar{z} + \left(\kappa\lambda^3 - \overline{\kappa}\frac{\epsilon^3}{\lambda^3}\right)t\right] \tag{1.6}$$

χ which satisfies to corresponding $\overline{\partial}$-problem or equivalently to following singular integral equation:

$$\chi(\lambda) = 1 + \int\int_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)} \int\int_C \chi(\mu, \overline{\mu}) R(\mu, \overline{\mu}; \lambda', \overline{\lambda'}) d\mu \wedge d\overline{\mu}, \tag{1.7}$$

here canonical normalization $\chi \to \chi_\infty = 1$ as $\lambda \to \infty$ of wave function is assumed and the kernel R by the formula [4, 22, 23, 24]

$$R(\mu, \overline{\mu}; \lambda, \bar{\lambda}; z, \bar{z}, t) = R_0(\mu, \overline{\mu}; \lambda, \bar{\lambda}) e^{F(\mu;z,\bar{z},t) - F(\lambda;z,\bar{z},t)} \tag{1.8}$$

is given.
Solutions $u(z,\bar{z},t)$ of VN equation are expressed via reconstruction formulas

$$u = -\epsilon + i\epsilon\chi_{1z} = -\epsilon - i\chi_{-1\bar{z}} \quad (1.9)$$

through the coefficients χ_1 and/or χ_{-1} of Taylor's series

$$\chi = \chi_0 + \chi_1\lambda + \chi_2\lambda^2 + \ldots, \quad \chi = \chi_\infty + \frac{\chi_{-1}}{\lambda} + \frac{\chi_{-2}}{\lambda^2} + \ldots \quad (1.10)$$

expansions in the neighborhoods of points $\lambda = 0$ and $\lambda = \infty$ of complex plane $\mathbb{C}$.
In constructing of exact solutions u of VN equation (1.1) two conditions must be satisfied [4, 20, 21, 22, 23, 24]: the condition of potentiality of operator L_1, or the absence in the first auxiliary linear problem (1.3) of the terms with first derivatives $u_1\partial_z$ and $u_2\partial_{\bar{z}}$, and the condition of reality $\bar{u} = u$ of solutions.

Introduction VI

The potentiality condition on operator L_1 in (1.3), or equivalently in terms of wave function χ the condition $\chi_0 = 1$ [4, 20, 21, 22, 23, 24], imposes severe restrictions on the kernel R_0 of $\bar{\partial}$-problem.
The condition of reality of solutions $\bar{u} = u$ leads to another following restriction on the kernel R_0 [20, 21, 23, 24, 22]:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\epsilon^3}{|\mu|^2 |\lambda|^2 \overline{\mu}\,\overline{\lambda}} \overline{R_0\left(-\frac{\epsilon}{\overline{\lambda}}, -\frac{\epsilon}{\lambda}, -\frac{\epsilon}{\overline{\mu}} - \frac{\epsilon}{\mu}\right)}, \qquad (1.11)$$

this restriction was obtained in the limit of "weak" fields.
Both conditions were successfully applied in calculations of broad classes of exact solutions of VN equation (1.1) by $\bar{\partial}$-dressing such as lumps [20, 21], solutions with functional parameters, multi-line solitons and plane wave type singular periodic solutions [22, 23, 24].

Introduction VII

In the present note we do not use the limit of weak fields (1.11) and impose the reality condition $u = \overline{u}$ directly to calculated complex solutions satisfying only to potentiality condition.

This approach makes it possible to receive besides multi-line soliton solutions also plane wave type singular periodic solutions (this was shown at first in [22, 24]) and their's superpositions.

The application of $\overline{\partial}$-dressing in the special limit of zero energy level has allowed for us to construct new exact solutions, nonstationary and stationary, of VN equation in the forms of linear superpositions of special line solitons and also linear superpositions of special plane wave type singular periodic solutions.

By construction these exact solutions represent also new exact transparent potentials of 2D Stationary Schrödinger equation and can find an applications as model potentials for electrons in planar structures of modern electronics.

Simple nonlinear superpositions of complex solutions I

The choice of delta-functional kernel

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = \pi \sum_n A_n \delta(\mu - M_n)\delta(\lambda - \Lambda_n) \tag{2.1}$$

with complex constant coefficients A_n and complex discrete spectral parameters $M_n \neq \Lambda_n$ leads to well known simple determinant formula [22, 24]

$$u = -\epsilon + \frac{\partial^2}{\partial z \partial \bar{z}} \ln \det A, \quad A_{lk} = \delta_{lk} + \frac{2iA_k}{M_l - \Lambda_k} e^{F(M_l) - F(\Lambda_k)} \tag{2.2}$$

for exact multi-line soliton and plane wave type singular periodic solutions of VN equation. The main problem in using this formula is satisfaction to reality and potentiality conditions.

Simple nonlinear superpositions of complex solutions II

It was shown in [22, 24] that the choice of kernel R_0 (2.1) in the form

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = \pi\sum_{m=1}^{N}\Big[a_m\lambda_m\delta(\mu-\mu_m)\delta(\lambda-\lambda_m)+a_m\mu_m\delta(\mu+\lambda_m)\delta(\lambda+\mu_m)\Big]$$

of N paired terms with discrete spectral parameters (μ_m,λ_m) allows to satisfy the potentiality condition $\chi_0 = 1$.

In the simplest cases $N = 1, 2$ one obtains from (2.1) – (2.3) the following expressions for $\det A$ [22, 24]:

$$N = 1 : \det A = \Big(1 + s_1 e^{\Delta F(\mu_1,\lambda_1)}\Big)^2, \qquad (2.3)$$

Simple nonlinear superpositions of complex solutions III

$$N=2:\det A=\left(1+s_1e^{\Delta F(\mu_1,\lambda_1)}+s_ne^{\Delta F(\mu_n,\lambda_n)}+we^{\Delta F(\mu_1,\lambda_1)+\Delta F(\mu_n,\lambda_n)}\right)^2. \tag{2.4}$$

Here, in (2.3) – (2.4) a_n, μ_n, λ_n $(n=1,\dots,N)$ are some complex constants; μ_n and λ_n also known as discrete spectral parameters which give spectral characterization for corresponding exact solutions.
The quantities s_n, w and $\Delta F(\mu_n,\lambda_n)$ are given by the formulas:

$$s_n:=ia_n\frac{\mu_n+\lambda_n}{\mu_n-\lambda_n};\quad \Delta F(\mu_n,\lambda_n):=F(\mu_n)-F(\lambda_n), \tag{2.5}$$

$$w:=-s_1s_n\cdot\frac{(\lambda_1-\lambda_n)(\lambda_n+\mu_1)(\mu_1-\mu_n)(\lambda_1+\mu_n)}{(\lambda_1+\lambda_n)(\lambda_n-\mu_1)(\mu_1+\mu_n)(\lambda_1-\mu_n)}. \tag{2.6}$$

The expression for $\det A$ in the case $N=2$ (2.4) is generated by two pairs of terms in (2.3) with discrete spectral variables (μ_1,λ_1) and (μ_n,λ_n).

Simple nonlinear superpositions of complex solutions IV

The formula for generally complex solution corresponding to one arbitrary pair $(\mu_m, \lambda_m), (m = 1, ..., N)$ of discrete spectral variables due to (2.1) – (2.3) and (2.5) has the form:

$$u^{(m)}(z, \bar{z}, t) = -\epsilon + \tilde{u}^{(m)}(z, \bar{z}, t) = -\epsilon - \epsilon \frac{2s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2} \tag{2.7}$$

It is remarkable that for $w = s_1 s_n$ in (2.4) for case $N = 2$ of two pairs of terms in (2.3) with spectral variables (μ_1, λ_1) and $(\mu_n, \lambda_n), (n = 2, ..., N)$, i. e. for the choice

$$\frac{(\lambda_1 - \lambda_n)(\lambda_n + \mu_1)(\mu_1 - \mu_n)(\lambda_1 + \mu_n)}{(\lambda_1 + \lambda_n)(\lambda_n - \mu_1)(\mu_1 + \mu_n)(\lambda_1 - \mu_n)} = -1, \tag{2.8}$$

which is equivalent to relation

$$(\lambda_1 - \mu_1)(\lambda_n - \mu_n)(\lambda_1\mu_1 + \lambda_n\mu_n) = 0, \quad n \neq 1, \tag{2.9}$$

the expression for $\det A$ (2.4) greatly simplifies

$$\det A = \left(1 + s_1 e^{\Delta F(\mu_1,\lambda_1)}\right)^2 \left(1 + s_n e^{\Delta F(\mu_n,\lambda_n)}\right)^2. \tag{2.10}$$

The solutions $\mu_1 = \lambda_1$ and $\mu_n = \lambda_n$ of (2.9) correspond to lumps (rationally decreasing at infinity exact solutions u [20, 21] of VN equation) and in accordance with $M_n \neq \Lambda_n$ in (2.1) will not be considered here, so it is assumed below that $\mu_m \neq \lambda_m$ in (2.3) for all terms $m = 1, \ldots, N$ with

Simple nonlinear superpositions of complex solutions VI

discrete spectral variables μ_m, λ_m. Under this requirement the relations (2.8), (2.9) reduce to more simple one:

$$\lambda_n \mu_n + \lambda_1 \mu_1 = 0, \quad n = 2, \ldots, N. \tag{2.11}$$

An application of general formulas (2.1) – (2.4) in the case $N = 2$ due to (2.8) or (2.11) leads to very simple expression for complex solution of VN equation

$$u(z, \bar{z}, t) = -\epsilon - 2\epsilon \sum_{m=1,n} \frac{s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2} \tag{2.12}$$

which is nonlinear superposition $u = \epsilon + u^{(1)} + u^{(n)}$ of two solutions $u^{(1)}$ and $u^{(n)}$ of the type (2.7) with corresponding pairs of spectral variables (μ_1, λ_1) and (μ_n, λ_n).

Simple nonlinear superpositions of complex solutions VII

Modulo ϵ the solution (2.12) is the sum of complex solutions $u^{(1)}$ and $u^{(n)}$. We have proved further that nonlinear superposition

$$
\begin{aligned}
u(z,\bar{z},t) &= -\epsilon + \tilde{u}^{(1)}(z,\bar{z},t) + \sum_{m=2}^{N} \tilde{u}^{(m)}(z,\bar{z}) = \\
&= -\epsilon - 2\epsilon \frac{s_1(\mu_1-\lambda_1)^2}{\mu_1\lambda_1} \frac{e^{\Delta F(\mu_1,\lambda_1)}}{(1+s_1 e^{\Delta F(\mu_1,\lambda_1)})^2} - \\
&-2\epsilon \sum_{m=2}^{N} \frac{s_m(\mu_m-\lambda_m)^2}{\mu_m\lambda_m} \frac{e^{\Delta F(\mu_m,\lambda_m)}}{(1+s_m e^{\Delta F(\mu_m,\lambda_m)})^2}
\end{aligned}
\tag{2.13}
$$

of arbitrary number $N > 2$ of complex solutions, solution $u^{(1)}(z,\bar{z},t) = -\epsilon+\tilde{u}^{(1)}(z,\bar{z},t)$ and $N-1 > 1$ solutions $u^{(m)}(z,\bar{z},t) = -\epsilon+\tilde{u}^{(m)}(z,\bar{z},t), m =$

Simple nonlinear superpositions of complex solutions VIII

$2, ... N$ of the type (2.7) is also exact solution of VN equation, when conditions (2.11) are fulfilled and parameters μ_1 and λ_1 are satisfied to additional restriction

$$\kappa_1 \lambda_1^3 - \kappa_2 \frac{\epsilon^3}{\mu_1^3} = 0. \tag{2.14}$$

Due to conditions (2.11), (2.14) the phases $\Delta F(\mu_n, \lambda_n)$ (2.5) in (2.13) take the forms:

$$\varphi_1(z, \bar{z}, t) := \Delta F(\mu_1, \lambda_1) = i \left[(\mu_1 - \lambda_1) z - \left(\frac{\epsilon}{\mu_1} - \frac{\epsilon}{\lambda_1} \right) \bar{z} \right. \\ \left. - 2 \left(\kappa_1 \lambda_1^3 - \kappa_2 \frac{\epsilon^3}{\lambda_1^3} \right) t \right], \tag{2.15}$$

SIMPLE NONLINEAR SUPERPOSITIONS OF COMPLEX SOLUTIONS IX

$$\varphi_m(z,\bar{z}) := \Delta F(\mu_m,\lambda_m) = i\left[(\mu_m-\lambda_m)z - \left(\frac{\epsilon}{\mu_m}-\frac{\epsilon}{\lambda_m}\right)\bar{z}\right], \qquad (2.16)$$

where $m = 2,\ldots,N$. These last expressions (2.15) and (2.16) mean that the first complex solution $u^{(1)}(z,\bar{z},t) = -\epsilon + \tilde{u}^{(1)}(z,\bar{z},t)$ of superposition (2.13) propagates with nonzero velocity on the plane (x,y) but all other complex solutions $u^{(m)}(z,\bar{z}) = -\epsilon + \tilde{u}^{(m)}(z,\bar{z}), (m = 2,...,N)$ of superposition (2.13) with $N > 2$ are fixed on the plane (x,y) stationary solutions of VN equation (1.1).

We have proved also that restrictions of the sum in (2.13) to every subsum of arbitrary terms $1 \leqslant i < i+1 < ... < j-1 < j \leqslant N$ due to conditions (2.11), (2.14) also lead to exact complex solutions

$$u(z,\bar{z},t) = -\epsilon + \sum_{n=i}^{j} \tilde{u}^{(n)} \qquad (2.17)$$

SIMPLE NONLINEAR SUPERPOSITIONS OF COMPLEX SOLUTIONS X

of VN equation. Complex solutions of VN equation given by (2.17) due to (2.15), (2.16) can be divided on two classes: the class of nonstationary solutions with $i \geq 1$ and class of stationary solutions with $i \geq 2$.

Linear superpositions of line soliton solutions I

For construction of real multi-line solitons via (2.2) besides potentiality condition satisfied by the kernel R_0 of the type (2.3) the reality condition $u = \overline{u}$ for solutions u must be fulfilled. This can be done choosing appropriately complex constants a_n and complex discrete spectral parameters (μ_n, λ_n) in (2.3) – (2.16) by several ways [24, 22].
For example, by imposing reality condition $u = \bar{u}$ on complex solutions (2.7),(2.12),(2.13) and (2.17) with additional assumption of real phases $\Delta F(\mu_n, \lambda_n) = \Delta\overline{F(\mu_n, \lambda_n)}$ (2.5) we have calculated real multi-line soliton solutions.
It was shown in the papers [22, 24] that to such real solutions u leads the following choice of parameters

$$a_n = -\overline{a}_n := ia_{n0}, \quad \mu_n = -\frac{\epsilon}{\overline{\lambda}_n}, \quad n = 1, \ldots, N \tag{3.1}$$

Linear superpositions of line soliton solutions II

with real constants a_{n0}. Due to (3.1) and under additional assumption of positive values of real constants s_n given by (2.5)

$$s_n = a_{n0}\frac{\lambda_n + \mu_n}{\lambda_n - \mu_n} \overset{\text{def}}{=} e^{\phi_{0n}} > 0, \quad n = 1, \ldots, N \tag{3.2}$$

the solution (2.7) takes the form of real nonsingular one-line soliton solution

$$u^{(n)} = -\epsilon + \tilde{u}^{(n)} = -\epsilon + \frac{|\lambda_n - \mu_n|^2}{2\cosh^2\frac{\varphi_n(x,y,t)+\phi_{0n}}{2}} \tag{3.3}$$

with real phases $\varphi_n(x, y, t)$ given by (2.5)

$$\varphi_n(x, y, t) := \Delta F(\mu_n, \lambda_n) = 2|\lambda_n|\left(1 + \frac{\epsilon}{|\lambda_n|^2}\right)(\vec{N}_n\vec{r} - V_n t), \tag{3.4}$$

Linear superpositions of line soliton solutions III

here $\vec{r} = (x, y)$, $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\varphi_n(x, y, t)$ and V_n are corresponding velocities of one-line solitons

$$\vec{N}_n = \left(\frac{\lambda_{nI}}{|\lambda_n|}, \frac{\lambda_{nR}}{|\lambda_n|}\right), \quad V_n = -\frac{1}{|\lambda_n|}\left(1 + \frac{\epsilon\left(\epsilon - |\lambda_n|^2\right)}{|\lambda_n|^4}\right)\mathrm{Im}(\kappa\lambda_n^3), \quad (3.5)$$

where $n = 1, \ldots, N$.
The conditions (2.11) for discrete spectral parameters (μ_n, λ_n), $(n > 1)$, in nonlinear superpositions (2.12) and (2.13) due to (3.1) lead to following parametrization of (μ_n, λ_n)

$$\mu_n = i\tau_n^{-1}\mu_1, \quad \lambda_n = i\tau_n\lambda_1, \quad (n = 2, \ldots, N) \quad (3.6)$$

with arbitrary real constants τ_n.

Linear superpositions of line soliton solutions IV

Nonsingular two-line soliton solution characterized by two pairs of discrete spectral variables (μ_1, λ_1) and (μ_2, λ_2) due to (2.12), (3.1), (3.2) and (3.6) takes the form

$$u(x,y,t) = -\epsilon + \sum_{n=1}^{2} \tilde{u}^{(n)} = -\epsilon + \sum_{n=1}^{2} \frac{|\lambda_n - \mu_n|^2}{2\cosh^2 \frac{\varphi_n(x,y,t)+\phi_{0n}}{2}}, \tag{3.7}$$

where $u^{(n)}(x,y,t) = -\epsilon + \tilde{u}^{(n)}$, $(n = 1,2)$ – one-line soliton solutions (3.3) and phases $\varphi_n(x,y,t), (n = 1,2)$ are given by (3.4).
It is evident due to expressions for vectors of normals (3.5) and parametrization (3.6) that solitons $u^{(1)}$ and $u^{(2)}$ in (3.7) move in the plane (x,y) perpendicularly to each other.

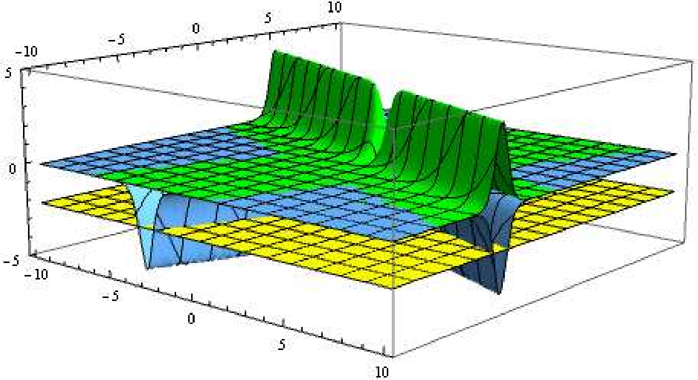


FIGURE 1: Potential $V_{\text{Schr}} = -2(u+\epsilon)$, corresponding of two-line solution (3.7) (blue), with energy level $E < 0$ (yellow plate) and squared absolute value of corresponding wave functions $|\psi(\mu_1)|^2 = |\psi(-\lambda_1)|^2$ (green).

Two soliton solution.

TwoSolitonSolution

It was shown in the paper [22, 24] that the limiting procedure for calculation of exact solutions u of VN equation with zero asymptotic values at infinity $u|_{|z|^2\to\infty} = -\epsilon \to 0$ can be defined by the following way

$$\epsilon \to 0, \quad \mu_n \to 0, \quad \frac{\epsilon}{\mu_n} \to -\overline{\lambda}_n \neq 0, \quad n = 1, \dots, N. \tag{3.8}$$

It is assumed that under procedure (3.8) the relations $\lambda_n = i\tau_n\lambda_1$ from (3.6) remain to be valid.
In the limit (3.8) two-line soliton solution (3.7) converts to linear superposition

$$u(x,y,t) = u^{(1)}_{\epsilon=0} + u^{(2)}_{\epsilon=0} = \sum_{n=1}^{2} \frac{|\lambda_n|^2}{2\cosh^2 \frac{\tilde{\varphi}_n(x,y,t)+\phi_{0n}}{2}} \tag{3.9}$$

of two one-line solitons $u^{(1)}_{\epsilon=0}$ and $u^{(2)}_{\epsilon=0}$

$$u^{(n)}_{\epsilon=0} = \frac{|\lambda_n|^2}{2\cosh^2 \frac{\tilde{\varphi}_n(x,y,t)+\phi_{0n}}{2}}, \quad n = 1, 2 \tag{3.10}$$

with phases $\tilde{\varphi}_n(x, y, t)$ given due to (3.4) and (3.8) by formulas

$$\tilde{\varphi}_n(x, y, t) = 2|\lambda_n|(\vec{N}_n\vec{r} - V_n t), \quad \vec{N}_n = \left(\frac{\lambda_{nI}}{|\lambda_n|}, \frac{\lambda_{nR}}{|\lambda_n|}\right), \quad n = 1, 2. \quad (3.11)$$

Here $\vec{r} = (x, y)$, $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x, y, t)$; V_n are corresponding velocities of one-line solitons

$$V_1 = -\frac{\mathrm{Im}(\kappa\lambda_1^3)}{|\lambda_1|}, \quad V_2 = -\frac{\mathrm{Im}(\kappa\lambda_2^3)}{|\lambda_2|} = \frac{\tau_2^3 \mathrm{Re}(\kappa\lambda_1^3)}{|\tau_2||\lambda_1|} \qquad (3.12)$$

derived by the use of (3.4), (3.6) and (3.8).
By special choice of spectral parameter λ_1 one of two one-line solitons $u^{(1)}_{\epsilon=0}$ or $u^{(2)}_{\epsilon=0}$ (not both) in linear superposition (3.9) can be "stopped". For example one can choose $V_2 = 0$, this achieves due to (3.12) for λ_1 satisfying to condition

$$\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3 = 0. \qquad (3.13)$$

The nonlinear superposition of the type (2.13) with $N \geqslant 3$ terms and parameters a_n, (μ_n, λ_n) and s_n satisfying to (2.14), (3.1), (3.2) and (3.6) takes the form

$$u = -\epsilon + \frac{|\lambda_1 - \mu_1|^2}{2\cosh^2 \frac{\varphi_1(x,y,t)+\phi_{01}}{2}} + \sum_{n=2}^{N} \frac{|\lambda_n - \mu_n|^2}{2\cosh^2 \frac{\varphi_n(x,y)+\phi_{0n}}{2}} \tag{3.14}$$

with phases φ_n (3.4) given due to (2.14) – (2.16) and (3.1), (3.2), (3.6) by expressions

$$\begin{aligned} \varphi_1(x,y,t) &= 2|\lambda_1| \left(1 + \frac{\epsilon}{|\lambda_1|^2}\right) (\vec{N}_1\vec{r} - V_1 t), \\ \varphi_n(x,y) &= 2|\lambda_1| \left(\tau_n + \frac{\epsilon}{\tau_n|\lambda_1|^2}\right) (\vec{N}_2\vec{r}). \end{aligned} \tag{3.15}$$

Here $n=2,\ldots,N$, $\vec{r}=(x,y)$; vectors of normals $\vec{N}_n$ and V_1 are given by following formulas

$$\vec{N}_1=\left(\frac{\lambda_{1I}}{|\lambda_1|},\frac{\lambda_{1R}}{|\lambda_1|}\right),\ \vec{N}_2=\left(\frac{\lambda_{1R}}{|\lambda_1|},-\frac{\lambda_{1I}}{|\lambda_1|}\right),\ V_1=\frac{i\kappa\lambda_1^3}{|\lambda_1|}\left(1+\frac{\epsilon\left(\epsilon-|\lambda_1|^2\right)}{|\lambda_1|^4}\right). \tag{3.16}$$

Due to (3.15) and (3.16) it follows that soliton $u^{(1)}=-\epsilon+\tilde{u}^{(1)}(x,y,t)$ moves on the plane (x,y) perpendicularly to others stationary solitons $u^{(n)}=-\epsilon+\tilde{u}^{(n)}(x,y)$, $(n=2,\ldots,N)$ with parallel lines of constant phases $\varphi_n(x,y)$.

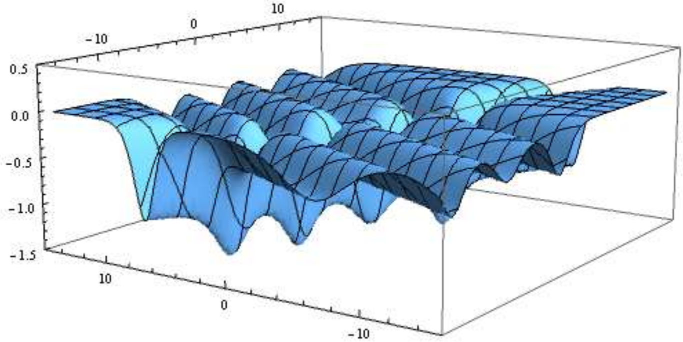


FIGURE 2: Potential $V_{\mathrm{Schr}} = -2(u + \epsilon)$ corresponding to five-line soliton solution ($N = 5$) (3.14).

Five-line soliton solution

FiveSolitonSolution

In the limit $\epsilon \to 0$ following to the rules (3.8) one obtains from (3.14) linear superposition of N one-line solitons

$$u = u^{(1)}_{\epsilon=0}(x,y,t) + \sum_{n=2}^{N} u^{(n)}_{\epsilon=0}(x,y) = \\ = \frac{|\lambda_1|^2}{2\cosh^2 \frac{\tilde{\varphi}_1(x,y,t)+\phi_{01}}{2}} + \sum_{n=2}^{N} \frac{|\lambda_n|^2}{2\cosh^2 \frac{\tilde{\varphi}_n(x,y)+\phi_{0n}}{2}} \tag{3.17}$$

with phases $\tilde{\varphi}_n$ (3.15) which in limit $\epsilon \to 0$ (3.8) are convert to the expressions

$$\tilde{\varphi}_1(x,y,t) = 2|\lambda_1|(\vec{N}_1\vec{r} - V_1 t), \quad \tilde{\varphi}_n(x,y) = 2\tau_n|\lambda_1|(\vec{N}_2\vec{r}), \quad n = 2,\ldots,N. \tag{3.18}$$

Here due to (3.6) and (3.11) vectors of normals $\vec{N}_{1,2}$ are given by (3.16), the value of velocity of nonstationary soliton due to (3.16) is $V_1 = \frac{i\kappa\lambda_1^3}{|\lambda_1|}$.

The first one-line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ from linear superposition (3.17) moves with velocity V_1 on the plane (x, y) perpendicularly to other $(N-1)$ stationary one-line solitons $u^{(n)}_{\epsilon=0}(x, y)$.
Remind that linear superposition (3.17) is obtained for $\epsilon \to 0$ by the use of limiting procedure (3.8), due to (2.14) and (3.6) it takes place under conditions

$$\lambda_n = i\tau_n\lambda_1, \quad \kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1^3} = 0, \quad n = 2, \ldots, N. \tag{3.19}$$

Evidently particular case of (3.9) with $V_2 = 0$ coincides due to (3.11) – (3.13) and (3.19) with the case $N = 2$ of linear superposition (3.17).
We have proved also that the subsums of arbitrary numbers of solitons $u^{(n)}_{\epsilon=0}$, $(n = 1, \ldots, N)$ from (3.17) are also solutions of VN equation. The set of such solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum) of line solitons and subset of stationary linear superpositions (without moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum) of stationary line solitons.

Linear superpositions of plane wave type periodic solutions I

By imposing reality condition $u = \bar{u}$ on complex solutions (2.7),(2.12),(2.13) and (2.17) with additional assumption of real phases $\Delta F(\mu_n, \lambda_n) = \overline{\Delta F(\mu_n, \lambda_n)}$ (2.5) multi-line soliton solutions are calculated in preceding section.
In contrast the reality condition $u = \bar{u}$ with assumption of pure imaginary phases $\Delta F(\mu_n, \lambda_n) = -\overline{\Delta F(\mu_n, \lambda_n)}$ (2.5) lead to plane wave type periodic solutions and their's superpositions.
Plane wave type solutions can be obtained by this way for example by the following choice of parameters $a_n, (\mu_n, \lambda_n)$ and s_n in (2.3)-(2.6) [24, 22]:

$$\mu_n = \frac{\epsilon}{\bar{\lambda}_n}, \; a_n = \left|\frac{\lambda_n - \mu_n}{\lambda_n + \mu_n}\right| e^{i \arg a_n}, \; s_n = -ie^{i \arg a_n} \mathrm{sign}\left(\frac{\lambda_n - \mu_n}{\lambda_n + \mu_n}\right), \; n = 1, \ldots \tag{4.1}$$

Linear superpositions of plane wave type periodic solutions II

Simple plane wave type periodic solution corresponding to one pair of spectral variables (μ_n, λ_n) due to (2.7) and (4.1) takes the form

$$u^{(n)} = -\epsilon + \tilde{u}^{(n)} = -\epsilon - \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y,t)+\arg a_n}{2} \mp \frac{\pi}{4}\right)} \qquad (4.2)$$

with real phases $\varphi_n(x, y, t) := -i\Delta F(\mu_n, \lambda_n)$

$$\varphi_n(x, y, t) := -i\Delta F(\mu_n, \lambda_n) = 2|\lambda_n|\left(\frac{\epsilon}{|\lambda_n|^2} - 1\right)(\vec{N}_n \vec{r} - V_n t). \qquad (4.3)$$

Linear superpositions of plane wave type periodic solutions III

here $\vec{r} = (x, y)$; $\vec{N}_n$ as unit vectors of normals to lines of constant values of phases $\varphi_n(x, y, t)$ and velocities V_n of periodic solutions are given by expressions:

$$\vec{N}_n = \left(\frac{\lambda_{nR}}{|\lambda_n|}, -\frac{\lambda_{nI}}{|\lambda_n|}\right), \; V_n = -\frac{1}{|\lambda_n|}\left(1 + \frac{\epsilon\left(\epsilon + |\lambda_n|^2\right)}{|\lambda_n|^4}\right)\mathrm{Re}(\kappa\lambda_n^3), \quad (4.4)$$

where $n = 1, \ldots, N$. Using (2.12) and (2.13) one can construct also nonlinear superpositions of simple wave type periodic solutions of the type (4.2).

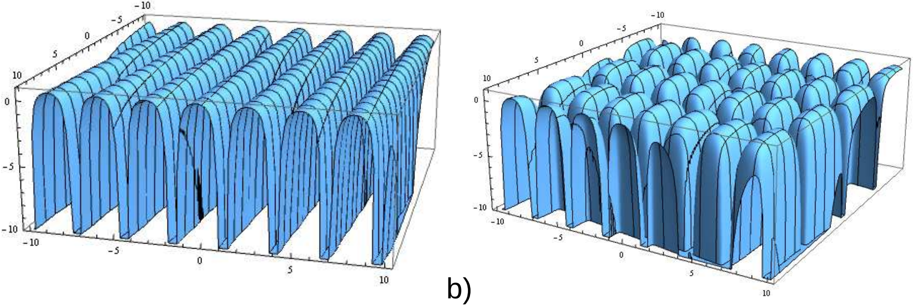
The conditions (2.11) for discrete spectral parameters (μ_n, λ_n), $(n > 1)$ in nonlinear superpositions (2.12) and (2.13) due to (4.1) in considered case lead to following parametrization of (μ_n, λ_n)

$$\lambda_n = i\tau_n\lambda_1, \quad \mu_n = i\tau_n^{-1}\mu_1, \quad (n = 2, \ldots, N) \quad (4.5)$$

with arbitrary real constants τ_n. Nonlinear superposition (2.12) of two simple plane wave type periodic solutions of the type (4.2) due to (4.1) and (4.5) takes the form

$$u(x,y,t) = -\epsilon + \sum_{n=1}^{2} \tilde{u}^{(n)} = -\epsilon - \sum_{n=1}^{2} \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y,t)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}, \quad (4.6)$$

where phases $\varphi_n(x,y,t)$ are given by (4.3). Due to expressions for vectors of normals (4.4) and parameterizations (4.5) it is evident that lines of constant values of phases $\varphi_n(x,y,t), (n=1,2)$ for $u^{(1)} = -\epsilon + \tilde{u}^{(1)}$ and $u^{(2)} = -\epsilon + \tilde{u}^{(2)}$ in (4.6) are moving perpendicularly to each other.



a) b)

FIGURE 3: a) Potential $\tilde{u}(x, y, t = 0) = u(x, y, t = 0) + \epsilon$ corresponding to simple plane wave type periodic solutions (4.2). b) Potential $\tilde{u}(x, y, t = 0) = u(x, y, t = 0) + \epsilon$, corresponding to nonlinear superposition of two simple plane wave type periodic solutions (4.6).

Simple plane wave type periodic solutions

PeriodicSolution

It was shown in the paper [22, 24] that the limiting procedure of calculation of exact solutions u of VN equation with zero values of parameter $\epsilon = 0$ defined by the following way

$$\epsilon \to 0, \quad \mu_n \to 0, \quad \frac{\epsilon}{\mu_n} \to \overline{\lambda}_n \neq 0, \quad n = 1, \dots, N \tag{4.7}$$

is applicable also in considered case of plane wave type solutions and their's superpositions. It is assumed that under procedure (4.7) the relations $\lambda_n = i\tau_n\lambda_1$ from (4.5) remain to be valid.
In the limit (4.7) nonlinear superposition (4.6) of two plane wave type periodic solutions converts to linear superposition

$$u(x, y, t) = u^{(1)}_{\epsilon=0} + u^{(2)}_{\epsilon=0} = -\sum_{n=1}^{2} \frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y,t)+\arg a_n}{2} + \frac{\pi}{4}\right)} \tag{4.8}$$

of two periodic solitons $u^{(1)}_{\epsilon=0}$ and $u^{(2)}_{\epsilon=0}$

$$u^{(n)}_{\epsilon=0} = -\frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y,t)+\arg a_n}{2} + \frac{\pi}{4}\right)}, \quad n = 1,2 \tag{4.9}$$

with phases $\tilde{\varphi}_n(x,y,t)$ and unit vectors $\vec{N}_n$ given due to (4.3) and (4.7) by formulas

$$\tilde{\varphi}_n(x,y,t) = -2|\lambda_n|(\vec{n}_n\vec{r} - V_n t), \quad \vec{N}_n = \left(\frac{\lambda_{nR}}{|\lambda_n|}, -\frac{\lambda_{nI}}{|\lambda_n|}\right), \quad n = 1,2, \tag{4.10}$$

here $\vec{r} = (x,y)$; $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x,y,t)$. The corresponding velocities V_n of simple plane wave type periodic solutions (4.9)

$$V_1 = -\frac{\mathrm{Re}(\kappa\lambda_1^3)}{|\lambda_1|}, \quad V_2 = -\frac{\mathrm{Re}(\kappa\lambda_2^3)}{|\lambda_2|} = \frac{\tau_2^3\mathrm{Im}(\kappa\lambda_1^3)}{|\tau_2||\lambda_1|} \tag{4.11}$$

are derived by the use of (4.3), (4.5) and (4.7). By special choice of spectral parameter λ_1 one of two of these periodic solutions $u^{(1)}_{\epsilon=0}$ or $u^{(2)}_{\epsilon=0}$ (not both) in linear superposition (4.8) can be "stopped". For example one can choose $V_2 = 0$, this achieves due to (4.11) for λ_1 satisfying to condition

$$\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3 = 0. \tag{4.12}$$

The nonlinear superposition of the type (2.13) with $N \geqslant 3$ terms and parameters a_n, (μ_n, λ_n), s_n satisfying to (2.14), (4.1) and (4.5) takes the form

$$u = -\epsilon - \frac{|\lambda_1 - \mu_1|^2}{2\cos^2\left(\frac{\varphi_1(x,y,t)+\arg a_1}{2} \mp \frac{\pi}{4}\right)} - \sum_{n=2}^{N} \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}. \tag{4.13}$$

Here phases φ_n (4.3) due to (2.14) – (2.16) and (4.1),(4.5), (4.7) are given by expressions

$$\begin{aligned} \varphi_1(x,y,t) &= 2|\lambda_1|\left(\frac{\epsilon}{|\lambda_1|^2}-1\right)(\vec{N}_1\vec{r}-V_1 t), \\ \varphi_n(x,y) &= 2|\lambda_1|\left(\frac{\epsilon}{\tau_n|\lambda_1|^2}-\tau_n\right)(\vec{N}_2\vec{r}), \end{aligned} \tag{4.14}$$

where $n=2,\ldots,N$, $\vec{r}=(x,y)$; unit vectors $\vec{N}_n$ and velocity V_1 are given by following formulas

$$\vec{N}_1=\left(\frac{\lambda_{1R}}{|\lambda_1|},-\frac{\lambda_{1I}}{|\lambda_1|}\right), \vec{N}_2=\left(-\frac{\lambda_{1I}}{|\lambda_1|},-\frac{\lambda_{1R}}{|\lambda_1|}\right), V_1=\frac{\kappa\lambda_1^3}{|\lambda_1|}\left(1+\frac{\epsilon\left(\epsilon+|\lambda_1|^2\right)}{|\lambda_1|^4}\right) \tag{4.15}$$

Due to (4.14) and (4.15) it follows that lines of constant values of phase $\varphi_1(x,y,t)$ of periodic solution $u^{(1)}$ move on plane (x,y) perpendicularly to parallel lines of constant phases $\varphi_n(x,y)$ of others stationary periodic solutions $u^{(n)}$, $(n=2,\ldots,N)$.

In the limit $\epsilon \to 0$ following to the rules (4.7) one obtains from (4.13) linear superposition of N simple plane wave type periodic solutions

$$u = u^{(1)}_{\epsilon=0} + \ldots + u^{(N)}_{\epsilon=0} = \frac{|\lambda_1|^2}{2\cos^2\left(\frac{\tilde{\varphi}_1(x,y,t)+\arg a_1}{2} \mp \frac{\pi}{4}\right)} + \\ + \sum_{n=2}^{N} \frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}, \tag{4.16}$$

here phases $\tilde{\varphi}_n$ are obtained from phases φ_n (4.14) by limiting procedure $\epsilon \to 0$ (4.7) and are given by expressions:

$$\tilde{\varphi}_1(x,y,t) = -2|\lambda_1|\left(\vec{N}_1\vec{r} - \frac{\kappa\lambda_1^3}{|\lambda_1|}t\right), \; \tilde{\varphi}_n(x,y) = -2\tau_n|\lambda_1|\left(\vec{N}_2\vec{r}\right), \tag{4.17}$$

Remind that linear superposition (4.16) is obtained for $\epsilon \to 0$ by the use of limiting procedure (4.7) due to (2.14) and (4.5), it takes place under conditions

$$\lambda_n = i\tau_n\lambda_1, \quad \kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1^3} = 0, \quad n = 2, \ldots, N. \tag{4.18}$$

The lines of constant values of phase $\varphi_1(x, y, t)$ of the first periodic solution $u^{(1)}_{\epsilon=0}$ from linear superposition (4.16) move with velocity $V_1 = \kappa\lambda_1^3/|\lambda_1|$ perpendicularly to lines of constant phases $\varphi_n(x, y)$ of $(N-1)$ other stationary periodic solutions $u^{(n)}_{\epsilon=0}, (n = 2, \ldots, N)$.
We have proved also that the subsums of arbitrary numbers of solutions $u^{(n)}_{\epsilon=0}$, $(n = 1, \ldots, N)$ from (4.16) are also solutions of VN equation. Evidently particular case of (4.8) with $V_2 = 0$ coincides due to (3.19) with the case $N = 2$ of linear superposition (4.16).
The set of constructed in present section solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum) and subset of stationary linear superpositions (without moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum).

Acknowledgements I

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