The Geometrical Approximations for the Rotating Shallow Water Equations

S.B. Medvedev
Institute of Computational Technologies SB RAS,
Novosibirsk
e-mail: medvedev@ict.nsc.ru

———————————————

4-8 June 2012, Novosibirsk
SOLITONS, COLLAPSES AND TURBULENCE:
Achievements, Developments and Perspectives

Surface of Earth is oblate spheroid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \tag{1}$$

where $b < a$.

Eccentricity e

$$e = \frac{\sqrt{a^2 - b^2}}{a}, \quad \alpha = \frac{a - b}{a},$$

$$e^2 \approx 1/150$$

.

Motion equations in rotating frame (λ, φ, r)

$$\frac{du}{dt} - \frac{uv}{r}\tan\varphi - 2\Omega\sin\varphi\, v + \frac{1}{\rho\cos\varphi}\frac{\partial p}{\partial\lambda} + \frac{1}{\cos\varphi}\frac{\partial\Phi}{\partial\lambda} = -\frac{uw}{r} - 2\Omega\cos\varphi\, w, \tag{2}$$

$$\frac{dv}{dt} + \frac{u^2}{r}\tan\varphi + 2\Omega\sin\varphi\, u + 2\Omega\cos\varphi\, w + \frac{1}{\rho r}\frac{\partial p}{\partial\varphi} + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} = -\frac{vw}{r}, \tag{3}$$

$$\frac{dw}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial\Phi}{\partial r} = \frac{u^2 + v^2}{2r} + 2\Omega\cos\varphi\, u, \tag{4}$$

Equations of continuity and advection of entropy:

$$\frac{d\rho}{dt} + \frac{\rho}{r\cos\varphi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\varphi)}{\partial\varphi}\right] + \rho\frac{\partial w}{\partial r} = -\frac{2w\rho}{r}, \tag{5}$$

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial s}{\partial\lambda} + \frac{v}{r}\frac{\partial s}{\partial\varphi} + w\frac{\partial s}{\partial r} = 0, \tag{6}$$

$$p = p(\rho, s), \quad u = r\cos\varphi\dot{\lambda}, \quad v = r\dot{\varphi}, \quad w = \dot{r}. \tag{7}$$

Approximations

1. Potential: $\Phi = \Phi_a - \frac{1}{2}\Omega^2 l^2 \approx gr$.

2. Traditional approximation: rhs=0

3. Shallow Atmosphere & Ocean: $\frac{1}{r} = \frac{1}{a+(r-a)} \approx \frac{1}{a}$

Approximation of metric form

Metric form for spherical coordinates

$$ds^2 = r^2 \cos^2 \varphi d\lambda^2 + r^2 d\varphi^2 + dr^2. \qquad (8)$$

Near spherical surface

$$r = a, \quad dr = dr. \qquad (9)$$

Approximate metric form

$$ds^2 = a^2 \cos^2 \varphi d\lambda^2 + a^2 d\varphi^2 + dr^2. \qquad (10)$$

Riemann manifold

$$ds^2 = \left(h_1 dq^1\right)^2 + \left(h_2 dq^2\right)^2 + \left(h_3 dq^3\right)^2, \tag{11}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \hat{J}\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = 0, \tag{12}$$

$$fg = (f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ...)(g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + ...) =$$

$$= f_0 g_0 + \varepsilon(f_0 g_1 + f_1 g_0) + \varepsilon^2(f_2 g_0 + f_1 g_1 + f_0 g_2) + ...$$

Rotating Shallow water equations

$$ds^2 = \left(h_1 dq^1\right)^2 + \left(h_2 dq^2\right)^2, \tag{13}$$

Lame coefficients

$$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}$$

Physical components of velocity $\mathbf{u}$

$$u_1 = h_1 \frac{dq^1}{dt} = h_1 \dot{q}^1, \quad u_2 = h_2 \frac{dq^2}{dt} = h_2 \dot{q}^2, \tag{14}$$

Equations for u_1, u_2, h

$$\frac{du_1}{dt} - \left[f + \frac{1}{h_1 h_2}\left(u_2\frac{\partial h_2}{\partial q^1} - u_1\frac{\partial h_1}{\partial q^2}\right)\right] u_2 + \frac{1}{h_1}\frac{\partial gh}{\partial q^1} = 0, \qquad (15)$$

$$\frac{du_2}{dt} + \left[f + \frac{1}{h_1 h_2}\left(u_2\frac{\partial h_2}{\partial q^1} - u_1\frac{\partial h_1}{\partial q^2}\right)\right] u_1 + \frac{1}{h_2}\frac{\partial gh}{\partial q^2} = 0, \qquad (16)$$

$$\frac{\partial (hD)}{\partial t} + \frac{\partial}{\partial q^1}\left(hD\frac{u_1}{H_1}\right) + \frac{\partial}{\partial q^2}\left(hD\frac{u_2}{H_2}\right) = 0, \qquad (17)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u_1}{h_1}\frac{\partial}{\partial q^1} + \frac{u_2}{h_2}\frac{\partial}{\partial q^2}, \qquad (18)$$

where $D = \det(g_{ij}) = h_1 h_2$, $f = f(q^1, q^2)$ - Coriolis function.

Equations for U_1, U_2, h

Covariant components

$$U_1 = h_1 u_1, \quad U_2 = h_2 u_2. \tag{19}$$

$$\frac{dU_1}{dt} + \frac{U_1^2}{2}\frac{\partial}{\partial q^1}\frac{1}{h_1^2} + U_2^2\frac{\partial}{\partial q^1}\frac{1}{2h_2^2} - fD\frac{U_2}{h_2^2} + \frac{\partial gh}{\partial q^1} = 0, \tag{20}$$

$$\frac{dU_2}{dt} + \frac{U_1^2}{2}\frac{\partial}{\partial q^2}\frac{1}{h_1^2} + U_2^2\frac{\partial}{\partial q^2}\frac{1}{2h_2^2} + fD\frac{U_1}{h_1^2} + \frac{\partial gh}{\partial q^2} = 0, \tag{21}$$

$$\frac{\partial(hD)}{\partial t} + \frac{\partial}{\partial q^1}\left(hD\frac{U_1}{h_1^2}\right) + \frac{\partial}{\partial q^2}\left(hD\frac{U_2}{h_2^2}\right) = 0. \tag{22}$$

Hamiltonian structure

$$\frac{\partial \mathbf{u}}{\partial t} + \hat{J}\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = 0, \qquad (23)$$

where Poisson bracket

$$\hat{J}(q^1, q^2; u_1, u_2, h) = \begin{pmatrix} 0 & -\frac{fD+\omega}{hD^2} & \frac{1}{h_1}\frac{\partial}{\partial q^1}\frac{1}{D} \\ \frac{fD+\omega}{hD^2} & 0 & \frac{1}{h_2}\frac{\partial}{\partial q^2}\frac{1}{D} \\ \frac{1}{D}\frac{\partial}{\partial q^1}\frac{1}{h_1} & \frac{1}{D}\frac{\partial}{\partial q^2}\frac{1}{h_2} & 0 \end{pmatrix}, \qquad (24)$$

$$\omega = \frac{\partial}{\partial q^1}(h_2 u_2) - \frac{\partial}{\partial q^2}(h_1 u_1)$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int \left[u_1^2 + u_2^2 + gh\right] hD\, dq^1 dq^2. \qquad (25)$$

For variables

$$U_1 = h_1 u_1, \quad U_2 = h_2 u_2, \quad H = hD, \tag{26}$$

Poisson bracket

$$\hat{J}(q^1, q^2; U_1, U_2, H) = \begin{pmatrix} 0 & -\frac{fD+\omega}{H} & \frac{\partial}{\partial q^1} \\ \frac{fD+\omega}{H} & 0 & \frac{\partial}{\partial q^2} \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & 0 \end{pmatrix}, \quad \omega = \frac{\partial U_2}{\partial q^1} - \frac{\partial U_1}{\partial q^2} \tag{27}$$

Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int \left[\frac{U_1^2}{h_1^2} + \frac{U_2^2}{h_2^2} + \frac{gH}{D} \right] h \, dq^1 dq^2, .$$

For plane geometry $h_1 = h_2 = 1$ and Coriolis parameter f therefore the difference is in term fD.

Formal transformation to eliminate f

$$V_1 = U_1 - \frac{\partial \xi}{\partial q^1}, \quad V_2 = U_2 - \frac{\partial \xi}{\partial q^2} + F(q^1, q^2), \tag{28}$$

where $\xi = \xi(q_1, q_2)$ arbitrary function and $F(q^1, q^2) = \int f(q^1, q^2) D \, dq^1$

$$\hat{J}(q^1, q^2; V_1, V_2, H) = \begin{pmatrix} 0 & -\frac{\Omega}{H} & \frac{\partial}{\partial q^1} \\ \frac{\Omega}{H} & 0 & \frac{\partial}{\partial q^2} \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & 0 \end{pmatrix}, \quad \Omega = \frac{\partial V_2}{\partial q^1} - \frac{\partial V_1}{\partial q^2} \tag{29}$$

$$\mathcal{H} = \frac{1}{2} \int \left[\frac{\left(V_1 + \frac{\partial \xi}{\partial q^1} \right)^2}{h_1^2} + \frac{\left(V_2 + \frac{\partial \xi}{\partial q^2} - F \right)^2}{h_2^2} + \frac{gH}{D} \right] H \, dq^1 dq^2. \tag{30}$$

Plane

$$\frac{du}{dt} - fv + g\frac{\partial h}{\partial x} = 0, \tag{31}$$

$$\frac{dv}{dt} + fu + g\frac{\partial h}{\partial y} = 0, \tag{32}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hu)}{\partial y} = 0, \tag{33}$$

$$f = f_0, \quad f = f_0 + \beta y$$

Casimir functional

$$\mathcal{C} = \int hC(q)dxdy, \quad q = \frac{1}{h}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right),$$

where $C(q)$ arbitrary function.

For general Hamiltonian

$$h\frac{\partial C}{\partial t} + \frac{\delta \mathcal{H}}{\delta u}\frac{\partial C}{\partial x} + \frac{\delta \mathcal{H}}{\delta v}\frac{\partial C}{\partial y} = 0 \tag{34}$$

or for RSWE Hamiltonian $\mathcal{H} = \frac{1}{2}\int \left[u^2 + v^2 + gh\right] h\, dxdy$

$$h\left(\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y}\right) = 0. \tag{35}$$

$C(q)$ is Lagrangian invariant.

General surface of rotation

Canonical metric form ($q_1 = \lambda$)

$$ds^2 = \left(h_1(q^2)dq^1\right)^2 + (dq^2)^2. \tag{36}$$

Isothermal coordinates

$$ds^2 = n^2(q^2)\left((dq^1)^2 + (dq^2)^2\right), \tag{37}$$

Volume-preserving coordinates:

$$ds^2 = (h_1(q_2)dq_1)^2 + (h_2(q_2)dq_2)^2, \quad h_1 h_2 = 1. \tag{38}$$

$$\frac{du_1}{dt} + \left[2\Omega_0 + \frac{u_1}{h_1}\right] \frac{u_2}{h_2}\frac{\partial h_1}{\partial q^2} + \frac{1}{h_1}\frac{\partial (gh)}{\partial q^1} = 0, \tag{39}$$

$$\frac{du_2}{dt} - \left[2\Omega_0 + \frac{u_1}{h_1}\right] \frac{u_1}{h_2}\frac{\partial h_1}{\partial q^2} + \frac{1}{h_2}\frac{\partial (gh)}{\partial q^2} = 0, \tag{40}$$

$$\frac{\partial h}{\partial t} + \frac{1}{h_1}\frac{\partial}{\partial q^1}(hu_1) + \frac{1}{h_1 h_2}\frac{\partial}{\partial q^2}(h_1 h u_2) = 0, \tag{41}$$

where Ω_0 angular velocity. Coriolis parameter

$$f = -2\Omega_0 \frac{1}{h_2}\frac{\partial h_1}{\partial q^2} \tag{42}$$

and the coefficient fD from Poisson bracket

$$fD = -\Omega_0 \frac{dh_1^2}{dq^2}. \tag{43}$$

Sphere

Radius a, longitude λ and latitude φ

$$ds^2 = a^2 \cos^2 \varphi \, d\lambda^2 + a^2 \, d\varphi^2, \quad f = 2\Omega_0 \sin \varphi \tag{44}$$

Volume-preserving coordinates: λ and $\mu = \sin \varphi$

$$ds^2 = a^2(1 - \mu^2) \, d\lambda^2 + \frac{a^2}{1 - \mu^2} \, d\mu^2, \quad f = 2\Omega_0 \mu \tag{45}$$

Isothermal coordinates: λ and $\theta = \operatorname{arctanh} \mu$

$$ds^2 = a^2(1 - \tanh^2 \theta) \left(d\lambda^2 + d\theta^2 \right), \quad f = 2\Omega_0 \tanh \theta \tag{46}$$

Spheroid

Radius a, longitude λ and reduced latitude ψ

$$ds^2 = a^2 \cos^2 \psi \, d\lambda^2 + a^2(1 - e^2 \cos^2 \psi) \, d\psi^2, \tag{47}$$

$$f = 2\Omega_0 \frac{\sin \psi}{\sqrt{1 - e^2 \cos^2 \psi}}, \tag{48}$$

$$fD = \Omega_0 \sin(2\psi). \tag{49}$$

From Spheroid to Sphere

$$\varepsilon(e, \psi) = \frac{1}{\sqrt{1 - e^2 \cos^2 \psi}} - 1 = \frac{1}{2} e^2 \cos^2 \psi + O(e^4 \cos^4 \psi). \quad (50)$$

$$ds^2 = a^2 \cos^2 \psi \, d\lambda^2 + \frac{a^2}{(1 + \varepsilon)^2} \, d\psi^2. \quad (51)$$

$$\frac{du}{dt} - (1+\varepsilon)\left[2\Omega_0 + \frac{1}{a\cos\psi}u\right] v\sin\psi + \frac{g}{a\cos\psi}\frac{\partial h}{\partial\lambda} = 0, \tag{52}$$

$$\frac{dv}{dt} + (1+\varepsilon)\left[2\Omega_0 + \frac{1}{a\cos\psi}u\right] u\sin\psi + (1+\varepsilon)\frac{g}{a}\frac{\partial h}{\partial\psi} = 0, \tag{53}$$

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\psi}\frac{\partial}{\partial\lambda}(hu) + (1+\varepsilon)\frac{1}{a\cos\psi}\frac{\partial}{\partial\psi}(hv\cos\psi) = 0, \tag{54}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\psi}\frac{\partial}{\partial\lambda} + (1+\varepsilon)\frac{v}{a}\frac{\partial}{\partial\psi}. \tag{55}$$

$$\mathcal{H} = \frac{1}{2}\int\left[u^2 + v^2 + gh\right] h\,\frac{a^2}{1+\varepsilon}\cos\psi\, d\lambda d\psi.$$

Hamiltonian expansion

$$U = u\,a\cos\psi, \quad V = v\frac{a}{1+\varepsilon}, \quad H = h\frac{a^2}{1+\varepsilon}\cos\psi, \tag{56}$$

$$\hat{J}(\lambda,\psi;U,V,H) = \begin{pmatrix} 0 & -q & \frac{\partial}{\partial\lambda} \\ q & 0 & \frac{\partial}{\partial\psi} \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial\psi} & 0 \end{pmatrix}, \tag{57}$$

where the potential vorticity

$$q = \frac{1}{H}\left(\Omega_0\sin(2\psi) + \frac{\partial V}{\partial\lambda} - \frac{\partial U}{\partial\psi}\right).$$

Hamiltonian

$$\mathcal{H} = \frac{1}{2a^2}\int\left[U^2\frac{1}{\cos^2\psi} + V^2(1+\varepsilon)^2 + gH\,\frac{1+\varepsilon}{\cos\psi}\right]H\,d\lambda d\psi$$

From Sphere to f-plane

Introduce new variable y such that $\varphi(0) = \varphi_0$.

$$ds^2 = a^2 \cos^2 \varphi d\lambda^2 + a^2 \left(\varphi'\right)^2 dy^2, \quad \varphi' = \frac{d\varphi}{dy}. \tag{58}$$

Coriolis parameter

$$f = 2\Omega_0 \sin \varphi(y). \tag{59}$$

Our Aim is to get a constant for fD

$$fD = -\Omega_0 \frac{d\left[a^2 \cos^2 \varphi(y)\right]}{dy} = f_0 D_0 \neq 0, \tag{60}$$

where $f_0 = f(\varphi_0)$ and $D_0 = D(\varphi_0)$.

$$\cos^2\varphi = \cos^2\varphi_0 - c_0 y, \qquad c_0 = \frac{f_0 D_0}{a^2\Omega_0}. \tag{61}$$

Solution

$$\varphi = \frac{\pi}{2} - \frac{1}{2}\arccos\left(c_0 y - \cos 2\varphi_0\right) \tag{62}$$

$$ds^2 = a^2\left(\cos^2\varphi_0 - c_0 y\right) d\lambda^2 + \frac{a^2 c_0^2}{4(\cos^2\varphi_0 - c_0 y)(\sin^2\varphi_0 + c_0 y)} dy^2. \tag{63}$$

$$U = ua\sqrt{\cos^2\varphi_0 - y}, \quad V = v\frac{a}{2\sqrt{(\cos^2\varphi_0 - y)(\sin^2\varphi_0 + y)}}, \tag{64}$$

$$H = h\frac{a^2}{2\sqrt{\sin^2\varphi_0 + y}} \tag{65}$$

Poisson bracket is identical to f-plane model

$$\hat{J}(\lambda, y; U, V, H) = \begin{pmatrix} 0 & -q & \frac{\partial}{\partial\lambda} \\ q & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial y} & 0 \end{pmatrix}, \tag{66}$$

$$q = \frac{1}{H}\left(\Omega_0 a^2 + \frac{\partial V}{\partial\lambda} - \frac{\partial U}{\partial y}\right).$$

Гамильтониан системы есть

$$\mathcal{H} = \frac{1}{2a^2}\int \left[\frac{U^2}{\cos^2\varphi_0 - y} + 4V^2(\cos^2\varphi_0 - y)(\sin^2\varphi_0 + y)\right] H\, d\lambda dy +$$

$$+\frac{1}{a^2}\int gH\,\sqrt{\sin^2\varphi_0 + y} H\, d\lambda dy.$$

Expansion for small y

$$\mathcal{H}_0 = \frac{1}{2a^2}\int \left[\frac{U^2}{\cos^2\varphi_0} + 4V^2\cos^2\varphi_0\sin^2\varphi_0 + 2gH\,\sin\varphi_0\right] H\, d\lambda dy, \tag{67}$$

$$\mathcal{H}_1 = \frac{1}{2a^2}\int y\left[\frac{U^2}{\cos^4\varphi_0} + 4V^2(\cos^2\varphi_0 - \sin^2\varphi_0) + \frac{gH}{\sin\varphi_0}\right] H\, d\lambda dy. \tag{68}$$

From Sphere to β-plane

Introduce dimensionless variables

$$l = (a/L)\lambda, \quad m = (a/L)\mu$$

, where L characteristic scale

$$ds^2 = L^2\left(1 - \frac{L^2m^2}{a^2}\right)dl^2 + L^2\left(1 - \frac{L^2m^2}{a^2}\right)^{-1}dm^2. \tag{69}$$

$$U = uL\sqrt{1 - \frac{L^2m^2}{a^2}}, \quad V = \frac{vL}{\sqrt{1 - \frac{L^2m^2}{a^2}}}, \tag{70}$$

Hamiltonian system

$$L\frac{\partial}{\partial t}\begin{pmatrix} U \\ V \\ \Phi \end{pmatrix} + \begin{pmatrix} 0 & -q & \frac{\partial}{\partial l} \\ q & 0 & \frac{\partial}{\partial m} \\ \frac{\partial}{\partial l} & \frac{\partial}{\partial m} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta U} \\ \frac{\delta \mathcal{H}}{\delta V} \\ \frac{\delta \mathcal{H}}{\delta h} \end{pmatrix} = 0, \qquad (71)$$

where potential vorticity

$$q = \frac{1}{h}\left(2\Omega_0 Lm\frac{L}{a} + \frac{\partial V}{\partial l} - \frac{\partial U}{\partial m}\right).$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int \left[U^2\left(1 - \frac{L^2m^2}{a^2}\right)^{-1} + V^2\left(1 - \frac{L^2m^2}{a^2}\right) + ghL^2\right] h\, dl dm.$$

Approximations may be obtained by expansion by small parameter L/a of the Hamiltonian $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

where

$$\mathcal{H}_0 = \frac{1}{2}\int \left[U^2 + V^2 + ghL^2\right] h\, dldm,$$

$$\mathcal{H}_1 = \frac{1}{2}\int \frac{L^2m^2}{a^2}\left[U^2 - V^2\right] h\, dldm.$$

Applications:

1. Numerics: conservation of Casimir functionals

2. Shallow flows over complex surfaces