Reduction Groups, Automorphic Lie algebras and integrable systems

A.Mikhailov

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- Classification of integrable equations - classification of Lax representations.

- Kac-Moody algebras: graded Lie algebras

- Automorphic Lie algebras: quasi–graded Lie algebras

- Finite reduction groups, automorphic Lie algebras and corresponding integrable systems

- soliton solutions for two-dimensional Volterra chain.

Lax pair:

$$L = \frac{d}{dx} - U(x,t;\lambda), \qquad A = \frac{d}{dt} - V(x,t;\lambda)$$

$$L\Psi = 0, \qquad A\Psi = 0$$

$$[L,A] = U_t - V_x + [U,V] = 0$$

Example (2-dimensional Volterra chain):

$$L = \partial_x + \lambda^{-1}\mathbf{u}\Delta - \lambda\Delta^{-1}\mathbf{u}$$

$$M = \partial_t + \lambda^{-1}\mathbf{a}\Delta - \lambda\Delta^{-1}\mathbf{a} + \lambda^{-2}\mathbf{b}\Delta^2 - \lambda^2\Delta^{-2}\mathbf{b}$$

where $\mathbf{u}, \mathbf{a}, \mathbf{b}$ are diagonal matrices and

$$\Delta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

In variables $\mathbf{u} = \text{diag}\,(\exp(\phi_i)), \mathbf{a} = \text{diag}\,(\theta_i \exp(\phi_i))$

$$\phi_{it} = \theta_{i,x} + \phi_{i,x}\theta_i + e^{2\phi_{i+1}} - e^{2\phi_{i-1}}, \qquad \theta_{i+1} - \theta_i + \phi_{i+1,\,x} + \phi_{i,\,x} = 0.$$

The field of rational functions $\mathbb{C}(\lambda)$ of λ
The ring of polynomials $\mathbb{C}[\lambda]$ in λ
The ring of Laurent polynomials $\mathbb{C}[\lambda, \lambda^{-1}]$.
$\mathcal{R}_\lambda(\Gamma)$ a ring of rational functions of λ with poles at $\lambda = \mu_k \in \Gamma = \{\mu_k \in \bar{\mathbb{C}}\}$ and with no other singularities.

$\mathbb{C}[\lambda] = \mathcal{R}_\lambda(\infty)$ and $\mathbb{C}[\lambda, \lambda^{-1}] = \mathcal{R}_\lambda(\infty, 0)$

Let $\mathfrak{A}$ be a simple Lie algebra over $\mathbb{C}$ ($\mathfrak{A} = sl(N, \mathbb{C})$):

$$\mathfrak{A}_\lambda(\Gamma) = \mathcal{R}_\lambda(\Gamma) \otimes_{\mathbb{C}} \mathfrak{A},$$

$$a(\lambda) = \sum_n \phi_n(\lambda) a_n \in \mathfrak{A}_\lambda(\Gamma), \quad a_n \in \mathfrak{A}, \ \phi_n(\lambda) \in \mathcal{R}_\lambda(\Gamma).$$

$$[\sum_n \phi_n(\lambda) a_n, \sum_m \psi_m(\lambda) a_m] = \sum_{n,m} \phi_n(\lambda) \psi_m(\lambda) [a_n, a_m]\,.$$

Kac-Moody algebras (V.Kac, 1968)

Let $\phi_1 : \mathfrak{A} \to \mathfrak{A}$ be an automorphism of a finite order n
$\Phi_1 : \mathfrak{A}_\lambda(0,\infty) \to \mathfrak{A}_\lambda(0,\infty)$, is defined as

$$\Phi_1(a(\lambda)) = \phi_1(a(\omega^{-1}\lambda))\,, \qquad \omega = \exp(\frac{2\pi i}{n})$$

$\Phi_1 \in \mathrm{Aut}\,\mathfrak{A}_\lambda(0,\infty)$, $\mathcal{G} = \langle \Phi_1\,;\, \Phi_1^n = \text{ id}\,\rangle \simeq \mathbb{Z}/n\mathbb{Z}$.

A Kac-Moody algebra $L(\mathfrak{A},\phi_1)$ can be defined as

$$L(\mathfrak{A},\phi_1) = \{a(\lambda) \in \mathfrak{A}_\lambda(0,\infty)\,|\,a(\lambda) = \phi_1(a(\omega^{-1}\lambda))\} \;.$$

$$L(\mathfrak{A},\phi_1) = \bigoplus_{k\in\mathbb{Z}} L^k(\mathfrak{A},\phi_1), \qquad [L^k(\mathfrak{A},\phi_1), L^m(\mathfrak{A},\phi_1)] \subset L^{k+m}(\mathfrak{A},\phi_1)\,,$$

where $L^k(\mathfrak{A},\phi_1) = \lambda^k\mathfrak{A}_k$ and $\mathfrak{A}_k = \{a \in \mathfrak{A}\,|\,\phi_1(a) = \omega^k a\}$.

$$A_n^1, ..., G_2^1, \quad A_n^2, D_n^2, E_6^2, \quad D_4^3.$$

Automorphic Lie algebras.

Example: The map $g_1 : \mathcal{R}_\lambda(0,\infty) \to \mathcal{R}_\lambda(0,\infty)$

$$g_1(\alpha(\lambda)) = \alpha(\omega^{-1}\lambda), \quad \alpha(\lambda) \in \mathcal{R}_\lambda(0,\infty)$$

is an automorphism of $\mathcal{R}_\lambda(0,\infty)$ of order n.

The ring $\mathcal{R}_\lambda(0,\infty) = \mathbb{C}[\lambda,\lambda^{-1}]$ has automorphism g_2 of order 2

$$g_2(\alpha(\lambda)) = \alpha(\lambda^{-1}), \quad \alpha(\lambda) \in \mathcal{R}_\lambda(0,\infty)\,.$$

Automorphisms g_1, g_2 generate a subgroup $G \subset \mathrm{Aut}\,\mathcal{R}_\lambda(0,\infty)$

$$G = \langle g_1, g_2\,;\, g_1^n = g_2^2 = g_1g_2g_1g_2 = \text{ id}\,\rangle \simeq \mathbb{D}_n\,.$$

Let $\phi_1, \phi_2 \in \mathrm{Aut}\,\mathfrak{A}$ and $\phi_1^n = \phi_2^2 = \phi_1\phi_2\phi_1\phi_2 = \mathrm{id}$.

$$\Phi_1(a(\lambda)) = \phi_1(a(\omega^{-1}\lambda))\,, \qquad \omega = \exp(\frac{2\pi i}{n})$$

$$\Phi_2(a(\lambda)) = \phi_2(a(\lambda^{-1}))\,, \quad \Phi_1, \Phi_2 \in \mathrm{Aut}\,\mathfrak{A}_\lambda(0,\infty).$$

$\mathcal{G} = \langle \Phi_1, \Phi_2\,;\, \Phi_1^n = \Phi_2^2 = \Phi_1\Phi_2\Phi_1\Phi_2 = \mathrm{id}\,\rangle \subset \mathrm{Aut}\,\mathfrak{A}_\lambda(0,\infty)$.
A subalgebra of $\mathfrak{A}_\lambda(0,\infty)$

$$\mathfrak{A}_\lambda^{\mathcal{G}}(0,\infty) = \{a(\lambda) \in \mathfrak{A}_\lambda(0,\infty) \,|\, a = \Phi_1(a) = \Phi_2(a)\}\ .$$

is an example of **automorphic Lie algebra**.

Finite reduction groups.

We will consider finite groups G whose elements are Möbius (fractional-linear) transformations. Every element $g \in G$ is represented by a transformation

$$\sigma_g(\lambda) = \frac{\alpha_g \lambda + \beta_g}{\gamma_g \lambda + \delta_g}, \qquad \alpha_g\delta_g - \beta_g\gamma_g \neq 0, \quad \alpha_g, \beta_g, \gamma_g, \delta_g \in \mathbb{C}. \tag{1}$$

Action of Möbius transformation on a rational function $f(\lambda) \in \mathbb{C}(\lambda)$ is defined as $\sigma_g : f(\lambda) \mapsto f(\sigma_g^{-1}(\lambda))$.

If the group average

$$\langle f(\lambda) \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(\sigma_g^{-1}(\lambda))$$

is not a constant, then $\langle f(\lambda) \rangle_G$ is a rational automorphic function.

According to F.Klein (1875), all finite subgroups of $PSL(2, \mathbb{C})$ are in the following list:

1. the additive group of integers modulo N, $\mathbb{Z}/N\mathbb{Z}$
2. the symmetry group of the dihedron with N vertices, $\mathbb{D}_N$
3. the symmetry group of the tetrahedron, $\mathbb{T}$
4. the symmetry group of the octahedron, $\mathbb{O}$
5. the symmetry group of the icosahedron, $\mathbb{I}$

Inner reduction group. $(\mathfrak{A} = \mathrm{sl}(N))$. $\mathrm{Aut}(sl(N))$:

$$\phi \in \mathrm{Aut}\, sl(N) \Rightarrow \phi(a) = \mathbf{Q} a \mathbf{Q}^{-1} \text{ or } \phi(a) = -\mathbf{Q} a^{tr} \mathbf{Q}^{-1}.$$

Let $\rho : G \mapsto \mathrm{PSL}(N, \mathbb{C})$ be a projective representation. We shall denote $\rho(g) = \mathbf{Q}_g$, where $\mathbf{Q}_g \in \mathrm{PSL}(N, \mathbb{C})$ is the corresponding $N \times N$ matrix.

With every element $g \in G$ we associate a pair

$$\Phi_g = (\sigma_g, \mathbf{Q}_g).$$

Obviously $\mathcal{G} = \{\Phi_g \,|\, g \in G\}$ is a group with multiplication $\Phi_g \Phi_h = \Phi_{gh}$.

The group $\mathcal{G}$ is called the (inner) **reduction group**, corresponding to a finite Möbius group G and representation ρ.

Automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ is defined as a $\mathcal{G}$-invariant subalgebra of $\mathfrak{A}_\lambda(\Gamma)$

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma) = \{a(\lambda) \in \mathfrak{A}_\lambda(\Gamma) \,|\, \Phi_g(a(\lambda)) = a(\lambda),\ \forall\, \Phi_g \in \mathcal{G}\}.$$

There is a natural projection $\mathcal{P}_{\mathcal{G}}$ of the linear space $\mathfrak{A}_\lambda$ onto $\mathfrak{A}_\lambda^{\mathcal{G}}$ given by the group average.

For $a(\lambda) \in \mathfrak{A}_\lambda$ we define $\mathcal{P}_{\mathcal{G}}(a(\lambda)) \in \mathfrak{A}_\lambda^{\mathcal{G}}$ as

$$\mathcal{P}_{\mathcal{G}}(a(\lambda)) = \langle a(\lambda)\rangle_{\mathcal{G}} = \frac{1}{|\mathcal{G}|}\sum_{\Phi\in\mathcal{G}} \Phi(a(\lambda)) = \frac{1}{|G|}\sum_{g\in G} \mathbf{Q}_g\, a(\sigma_g^{-1}(\lambda))\mathbf{Q}_g^{-1}\,.$$

Obviously $\mathcal{P}_{\mathcal{G}}^2 = \mathcal{P}_{\mathcal{G}}$.

The projection $\mathcal{P}_{\mathcal{G}} : \mathfrak{A}_\lambda \mapsto \mathfrak{A}_\lambda^{\mathcal{G}}$ is a surjective linear map, but it is *not* a Lie algebra homomorphism.

Orbits of G. For any $\gamma_0 \in \bar{\mathbb{C}}$ we denote

- the **orbit** $G(\gamma_0) = \{g(\gamma_0) \,|\, g \in G\}$.

- the **isotropy subgroup** $G_{\gamma_0} = \{g \in G \,|\, g(\gamma_0) = \gamma_0\}$.

- If the group G_{γ_0} is nontrivial, i.e. $|G_{\gamma_0}| = n > 1$, then γ_0 is a **fixed point** of order n.

- If γ_0 is a fixed point of order $n > 1$, than the orbit $G(\gamma_0)$ is a **degenerated orbit of degree** n. $|G(\gamma_0)| = |G|/|G_{\gamma_0}|$. Orbits corresponding to generic points we call **generic**.

Let $\mathfrak{A} = \mathrm{sl}(N)$. To construct $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ we choose:

- A finite Möbius group G, represented by Möbius transformations σ_g

- A finite set of points $\hat{\gamma} = \{\gamma_k \in \bar{\mathbb{C}}\}$ and define its orbit $\mathbf{\Gamma} = \cup G(\gamma_k)$

- A projective representation $\rho : G \mapsto \mathrm{PSL}(N, \mathbb{C})$, so that $\rho(g) = \mathbf{Q}_g$

Then $\mathcal{G} = \{(\sigma_g, \mathbf{Q}_g) \,|\, g \in G\}$ and

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma}) = \left\{ \frac{1}{|G|} \sum_{g \in G} \frac{\mathbf{Q}_g a \mathbf{Q}_g^{-1}}{(\sigma_g^{-1}(\lambda) - \gamma)^n} \,|\, a \in \mathfrak{A},\ \gamma \in \hat{\gamma},\ n \in \mathbb{N}_0 \right\}$$

Example: In $\mathfrak{A} = sl(2, \mathbb{C}), G = \mathbb{D}_2$. We represent the group G by the Möbius transformations

$$g_1(\lambda) = -\lambda, \qquad g_2(\lambda) = \lambda^{-1}.$$

There are orbits

$$\Gamma_0 = \{0, \infty\}, \quad \Gamma_1 = \{\pm 1\}, \quad \Gamma_i = \{\pm i\}, \quad \Gamma_\mu = \{\pm\mu, \pm\mu^{-1}\}.$$

The reduction group $\mathcal{G} \sim \mathbb{D}_2$ is generated by

$$\Phi_1(a(\lambda)) = \mathrm{s}_3 a(-\lambda)\mathrm{s}_3, \qquad \Phi_2(a(\lambda)) = \mathrm{s}_1 a(\lambda^{-1})\mathrm{s}_1.$$

$$\mathrm{s}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathrm{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathrm{s}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \mathrm{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $sl(2,\mathbb{C})$ we take standard basis $\mathbf{e},\mathbf{f},\mathbf{h}$

$$\mathbf{e}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ \mathbf{f}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\mathbf{h}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Lie algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ is generated by

$$\mathbf{e}^1=2\langle\lambda\mathbf{e}\rangle_{\mathcal{G}},\ \mathbf{f}^1=2\langle\lambda\mathbf{f}\rangle_{\mathcal{G}},\ \mathbf{h}^2=2\langle\lambda^2\mathbf{h}\rangle_{\mathcal{G}}$$

Evaluating the group average we get:

$$\mathbf{e}^1=\begin{pmatrix}0&\lambda\\\lambda^{-1}&0\end{pmatrix},\quad \mathbf{f}^1=\begin{pmatrix}0&\lambda^{-1}\\\lambda&0\end{pmatrix},\quad \mathbf{h}^2=(\lambda^2-\lambda^{-2})\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Their commutators are ($J=\lambda^2+\lambda^{-2}$):

$$[\mathbf{e}^1,\mathbf{f}^1]=\mathbf{h}^2,\ [\mathbf{h}^2,\mathbf{e}^1]=2J\mathbf{e}^1-4\mathbf{f}^1,\ [\mathbf{h}^2,\mathbf{f}^1]=-2J\mathbf{f}^1+4\mathbf{e}^1.$$

$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ has basis : $\quad A=\bigcup_{n\in\mathbb{N}} A_n\,,\qquad A_n=\{J^{n-1}\mathbf{e}^1,\ J^{n-1}\mathbf{f}^1,\ J^{n-1}\mathbf{h}^2\}$

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)=\bigoplus_{k=1}^{\infty}\mathcal{A}^k,\quad [\mathcal{A}^p,\mathcal{A}^q]\subset\mathcal{A}^{p+q}\bigoplus\mathcal{A}^{p+q-1},\quad \mathcal{A}^n=\mathrm{Span}_{\mathbb{C}}A_n.$$

Proposition 1. *Let* $\mathfrak{A} = sl(2,\mathbb{C})$ *and* $\mathcal{G} \simeq \mathbb{D}_2$*. Automorphic Lie algebras* $\mathfrak{A}_\lambda^{\mathbb{D}_2}(\Gamma_0)$*,* $\mathfrak{A}_\lambda^{\mathbb{D}_2}(\Gamma_1)$ *and* $\mathfrak{A}_\lambda^{\mathbb{D}_2}(\Gamma_i)$*, corresponding to degenerated orbits, are graiding–isomorphic.*

Proposition 2. *Let* $\mathfrak{A} = sl(2,\mathbb{C})$ *and* $\mathcal{G} \simeq \mathbb{D}_2$*. Automorphic Lie algebras corresponding to generic and degenerated orbits are not isomorphic.*

Theorem 1. *Let* $\mathfrak{A} = sl(2,\mathbb{C})$*,* $\mathcal{G}$ *be any finite non-cyclic reduction group and* Γ *be any degenerated orbit of the corresponding Möbius group. Then the automorphic Lie algebra* $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ *is grading isomorphic to* $\mathfrak{A}_\lambda^{\mathbb{D}_2}(\Gamma_0)$*.*

The Theorem has been proven by: R.Bury, A.Mikhailov and independently by: S.Lombardo, J.Sanders

$sl(2)$ **Automorphic Lie algebras with a finite reduction group**

$\mathcal{A}^0$ the polynomial part of the Loop algebra $\mathfrak{A}_\lambda(\infty) = \mathbb{C}[\lambda] \otimes_{\mathbb{C}} sl(2, \mathbb{C})$, when the reduction group is trivial;

$\mathcal{A}^1$ the subalgebra $L_+(\mathfrak{A}, \phi), \ \phi^2 = id$ of the Kac-Moody algebra, this case corresponds to $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ with $\mathcal{G} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\Gamma = \{\infty\}$;

$\mathcal{A}_\mu^1$ algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ with $\mathcal{G} \simeq \mathbb{Z}/2\mathbb{Z}$ and a generic orbit $\Gamma_\mu = \{\pm\mu\}$;

$\mathcal{A}^2$ algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a degenerated orbit $\Gamma = \{0, \infty\}$;

$\mathcal{A}_\mu^2$ algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a generic orbit $\Gamma_\mu = \{\pm\mu, \pm\mu^{-1}\}$.

Integrable equations corresponding to $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$, $\mathfrak{A} = sl(2)$.

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma) = \bigoplus_{k=1}^{\infty} \mathcal{A}^k, \quad \mathcal{A}^n = J^{n-1}\mathrm{Span}_{\mathbb{C}}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}.$$

$J = J(\lambda)$ - automorphic function, and $\mathbf{a}_k = \mathbf{a}_k(\lambda)$.

We take the Lax pair (L, M) of the form

$$L = \partial_x + \sum_{i=1}^{3} u_i(x,t)\mathbf{a}_i$$

$$M = \partial_t + \sum_{i=1}^{3} v_i(x,t)\mathbf{a}_i + \sum_{i=1}^{3} w_i(x,t)J\mathbf{a}_i$$

The *compatibility condition* of the Lax pair defines a nonlinear integrable system for the entries of $\mathbf{X}$ and $\mathbf{T}$

$$\mathbf{T}_x - \mathbf{X}_t + [\mathbf{X}, \mathbf{T}] = 0$$

$\mathcal{A}^0$ the polynomial part of the Loop algebra $\mathfrak{A}_\lambda(\infty) = \mathbb{C}[\lambda] \otimes_{\mathbb{C}} sl(2, \mathbb{C})$, trivial reduction group:

The NLS:

$$u_t = u_{xx} + 2vu^2,$$

$$-v_t = v_{xx} + 2uv^2$$

or (gauge equivalent) the Heisenberg model

$$u_t = u_{xx} - \frac{2u_x^2}{u - v},$$

$$-v_t = v_{xx} - \frac{2v_x^2}{v - u}.$$

$\mathcal{A}^1$ the $L_+(\mathfrak{A}, \phi)$ subalgebra of the Kac-Moody algebra: $\mathfrak{A}^{\mathcal{G}}_{\lambda}(\Gamma)$ with $\mathcal{G} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\Gamma = \{\infty\}$;

The derivative NLS

$$u_{1t} = -\frac{1}{2}(u_1^2 u_2)_x - \frac{1}{2}u_{1xx}$$

$$u_{2t} = -\frac{1}{2}(u_1 u_2^2)_x + \frac{1}{2}u_{2xx}$$

$\mathcal{A}^2$ algebra $\mathfrak{A}^{\mathcal{G}}_{\lambda}(\Gamma)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a degenerated orbit $\Gamma = \{0, \infty\}$:

$$u_{1t} = -\frac{1}{2}(u_1^2 u_2)_x - \frac{1}{2}u_{1xx} + 2u_{2x}$$

$$u_{2t} = -\frac{1}{2}(u_1 u_2^2)_x + \frac{1}{2}u_{2xx} + 2u_{1x}$$

For the generic orbits and all groups we obtain systems of the form:

$$
\begin{aligned}
u_t &= u_{xx} - \frac{2u_x^2}{u-v} - \frac{2}{(u-v)^2}[P(u,v)u_x - R(u)v_x] \\
-v_t &= v_{xx} + \frac{2v_x^2}{v-u} + \frac{2}{(u-v)^2}[P(u,v)v_x - R(v)u_x]
\end{aligned}
$$

where

$$
\begin{aligned}
&P(u,v) = 2au^2v^2 + b(uv^2 + vu^2) + 2cuv + d(u+v) + 2e \\
&R(u) = au^4 + bu^3 + cu^2 + du + e.
\end{aligned}
$$

$\mathcal{A}_\mu^0$ (Trivial reduction group): $P(u,v) = R(u) = 0.$

$\mathcal{A}_\mu^1$ algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ with $\mathcal{G} \simeq \mathbb{Z}/2\mathbb{Z}$ and a generic orbit $\Gamma_\mu = \{\pm\mu\}$:

$$P(u,v) = 2\mu uv, \qquad R(u) = \mu u^2.$$

$\mathcal{A}_\mu^2$ algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a generic orbit $\Gamma_\mu = \{\pm\mu, \pm\mu^{-1}\}$:

$$P(u,v) = \frac{2\mu}{\mu^4 - 1}(u^2v^2 - (\mu^2 + \mu^{-2})uv + 1),$$

$$R(u) = \frac{\mu}{\mu^4 - 1}(u^4 - (\mu^2 + \mu^{-2})u^2 + 1)$$

Only the groups $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$ have **irreducible faithful projective** representations of dimension 3 or higher.

1. $\mathbb{T}$ has one 2-d and one 3-d representations
2. $\mathbb{O}$ has one 2-d, one 3-d and one 4-d representations
3. $\mathbb{I}$ has two 2-d, two 3-d, two 4-d, one 5-d and one 6-d representations.

$\mathfrak{A} = sl(3,\mathbb{C})$. Tetrahedral group (the same equation for Octahedral and Icosahedral groups!):

$$\sigma_s(\lambda) = \omega\lambda, \quad \sigma_r(\lambda) = \frac{\lambda+2}{\lambda-1},, \qquad \omega = \exp\left(\frac{2\pi i}{3}\right)$$

The group $\mathbb{T}$ has a 3-dimensional irreducible faithful projective representation $\rho : \mathbb{T} \mapsto \mathrm{PSL}(3,\mathbb{C})$ with elements s and r represented by:

$$\mathbf{Q}_{\mathbb{T}s} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{Q}_{\mathbb{T}r} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

The reduction group $\mathcal{G} \sim \mathbb{T}$ generated by two elements $\Phi_{\mathbb{T}s} = (\sigma_s, \mathbf{Q}_{\mathbb{T}s})$, $\Phi_{\mathbb{T}r} = (\sigma_r, \mathbf{Q}_{\mathbb{T}r})$.

$\mathfrak{A} = sl(3, \mathbb{C}),\ G = \mathbb{T}$:

$$\mathbf{a}_1 = \langle \lambda \mathbf{e}_{13} \rangle_{\mathbb{T}}, \mathbf{a}_2 = \langle \lambda \mathbf{e}_{21} \rangle_{\mathbb{T}}, \mathbf{a}_3 = \langle \lambda \mathbf{e}_{32} \rangle_{\mathbb{T}},\ \ J = \left\langle \lambda^3 \right\rangle_{\mathbb{T}}$$

$$L = \partial_x + \sum_{i \in \mathbb{Z}_3} u_i(x,t)\mathbf{a}_i$$

$$u_t = u_{xx} + v_x^2 + \theta_u v_x$$

$$-v_t = v_{xx} + u_x^2 - \theta_v u_x$$

where

$$\theta = a e^{-(u+v)} + a_1 e^{-\omega u - \omega^* v} + a_2 e^{-\omega^* u - \omega v}, \quad \omega = e^{\frac{2\pi i}{3}}$$

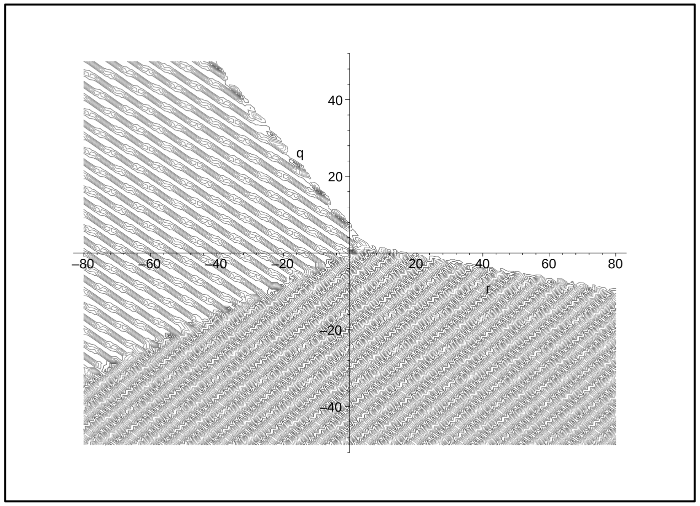
$\mathfrak{A} = sl(4, \mathbb{C})$, $G = \mathbb{O}$ (the same equation for the icosahedral group!):

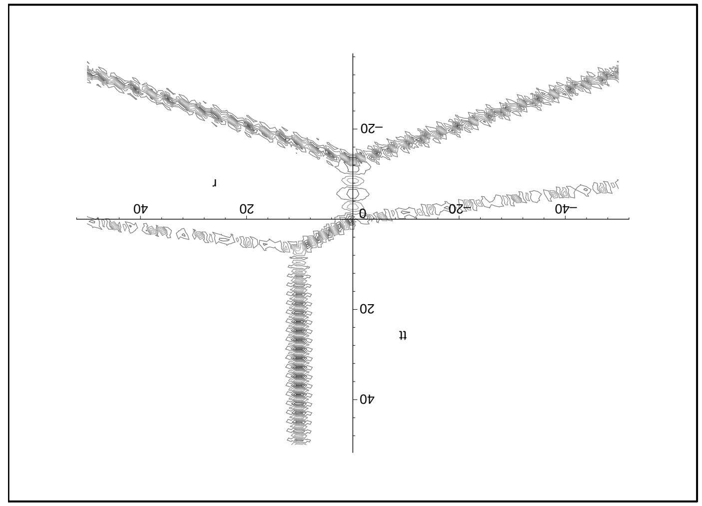
$$\psi_{1,t} = i(\psi_{2,x}^2 - \psi_{3,x}^2) \\ + (\psi_{2,x} + \psi_{3,x})e^{\psi_1+\psi_2+\psi_3} + (\psi_{2,x} + \psi_{3,x})e^{\psi_1-\psi_2-\psi_3} \\ + (\psi_{2,x} + \psi_{3,x})e^{-\psi_1+i\psi_2-i\psi_3} + (\psi_{2,x} + \psi_{3,x})e^{-\psi_1-i\psi_2+i\psi_3}$$

$$\psi_{2,t} = i\psi_{2,xx} - i\psi_{1,x}\psi_{3,x} \\ + (\psi_{1,x} + \psi_{3,x})e^{\psi_1+\psi_2+\psi_3} + (\psi_{1,x} - \psi_{3,x})e^{\psi_1-\psi_2-\psi_3} \\ + i\psi_{3,x}e^{-\psi_1-i\psi_2+i\psi_3} - i\psi_{3,x}e^{-\psi_1+i\psi_2-i\psi_3}$$

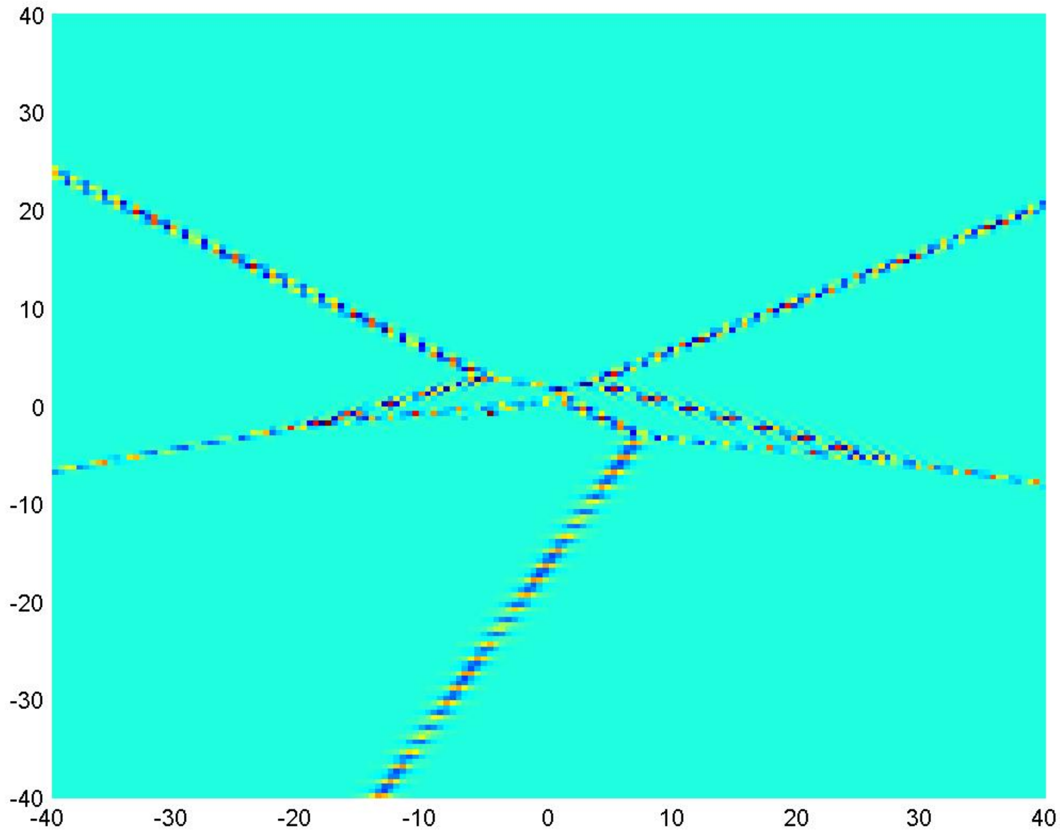
$$\psi_{3,t} = -i\psi_{3,xx} + i\psi_{1,x}\psi_{2,x} \\ + (\psi_{1,x} + \psi_{2,x})e^{\psi_1+\psi_2+\psi_3} + (\psi_{1,x} - \psi_{2,x})e^{\psi_1-\psi_2-\psi_3} \\ - i\psi_{2,x}e^{-\psi_1-i\psi_2+i\psi_3} + i\psi_{2,x}e^{-\psi_1+i\psi_2-i\psi_3}$$

Two-dimensional Volterra system. Soliton solutions





$$N = 5$$



$N = 5$

$1+1$–dimensional Volterra system (AVM 1979):

$$\phi_{it} = \theta_{i,x} + \phi_{i,x}\theta_i + e^{2\phi_{i+1}} - e^{2\phi_{i-1}}\,, \qquad \phi_{i+N} = \phi_i$$

$$\theta_{i+1} - \theta_i + \phi_{i+1,\,x} + \phi_{i,\,x} = 0\,, \qquad \sum_{i=1}^{N} \phi_i = \mathrm{const} = 0\,.$$

Continuous limit $\rightarrow$ KP equation ($N \rightarrow \infty,\ Nh = 1$):

$$\phi_i(x,t) = h^2 u(\xi,\eta,\tau), \quad h = \frac{L}{N} \rightarrow 0,$$

$$\tau = h^3 t, \quad \xi = ih + 4ht, \quad \eta = h^2 x$$

$$u_\tau = \frac{2}{3}u_{\xi\xi\xi} + 8uu_\xi - 2D_\xi^{-1}u_{\eta\eta} + O(h^2).$$

By a linear transformation (Fourier transform):

$$\phi_n = \sum_{k=1}^{N-1} \omega^{kn} \chi_k\,, \quad \omega = \exp(2\pi i/N)$$

we can diagonalise the linear part of the system

$$\chi_{kt} = \frac{1+\omega^k}{1-\omega^k} \chi_{kxx} + (\omega^k - (\omega^k)^*)\chi_k + \ldots, \quad k = 1, \ldots N-1$$

or

$$\chi_{kt} = i \cot(\frac{\pi k}{N}) \chi_{kxx} + 2i \sin(\frac{2\pi k}{N}) \chi_k + \ldots, \quad k = 1, \ldots N-1$$

If ϕ_n are real, then $\chi_k = \chi^*_{-k}$.

For $N = 3$, $\omega = \exp(2\pi i/3)$:

$$i u_t = u_{xx} + (u_x^*)^2 + e^{-2u-2u^*} + \omega^* e^{-2\omega u - 2\omega^* u^*} + \omega e^{-2\omega^* u - 2\omega u^*}$$

D_N invariant Lax pair:

$$L(\lambda) = \mathbf{S}L(\omega^{-1}\lambda)\mathbf{S}^{-1}, L(\lambda) = -L^A(\lambda^{-1}) :$$

$$L = \partial_x + \lambda^{-1}\mathbf{u}\boldsymbol{\Delta} - \lambda\boldsymbol{\Delta}^{-1}\mathbf{u}$$

$$M = \partial_t + \lambda^{-1}\mathbf{a}\boldsymbol{\Delta} - \lambda\boldsymbol{\Delta}^{-1}\mathbf{a} + \lambda^{-2}\mathbf{u}\boldsymbol{\Delta}\mathbf{u}\boldsymbol{\Delta} - \lambda^2\boldsymbol{\Delta}^{-1}\mathbf{u}\boldsymbol{\Delta}^{-1}\mathbf{u}$$

with

$$\mathbf{S} = \begin{pmatrix} \omega & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \omega^2 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \omega^{N-1} & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

and $\mathbf{u} = \text{diag}\,(\exp(\phi_i)), \mathbf{a} = \text{diag}\,(\theta_i \exp(\phi_i))$.

$[L, M] = 0 \Longleftrightarrow$ there exist a fundamental solution $\psi(\lambda, x, t)$

$$L\psi = 0, \quad M\psi = 0.$$

The Lax pair, (L_0, M_0), corresponding to the trivial solution $\phi_i \equiv 0$ is given by

$$L_0 = \partial_x + \lambda^{-1}\Delta - \lambda\Delta^{-1}$$

$$M_0 = \partial_t + \lambda^{-2}\Delta^2 - \lambda^2\Delta^{-2}$$

and the solution, ψ_0 to

$$L_0\psi_0 = 0, \quad M_0\psi_0 = 0$$

is given by

$$\psi_0 = e^{(-\lambda^{-1}\Delta+\lambda\Delta^{-1})x+(-\lambda^{-2}\Delta^2+\lambda^2\Delta^{-2})t}$$

Morover, it follows from the reduction group that ψ and ψ_0 satisfies the conditions

$$\begin{array}{lll} 1. & \mathbf{S}\psi(\omega^{-1}\lambda)\mathbf{S}^{-1} = \psi(\lambda), & \mathbf{S}\psi_0(\omega^{-1}\lambda)\mathbf{S}^{-1} = \psi_0(\lambda), \\ 2. & [\psi^{-1}(\lambda^{-1})]^T = \psi(\lambda), & [\psi_0^{-1}(\lambda^{-1})]^T = \psi_0(\lambda). \end{array}$$

If $\phi_i(x,t)$ are real, then we also have

$$3. \quad \psi^*(\lambda^*) = \psi(\lambda), \qquad \psi_0^*(\lambda^*) = \psi_0(\lambda).$$

Rational dressing.

We represent ψ in the form $\psi(x,t,\lambda) = \chi(x,t,\lambda)\psi_0(x,t,\lambda)$ and $L\psi = 0$ then

$$L = \chi(x,t,\lambda) L_0 \chi^{-1}(x,t,\lambda)$$

Let us assume that χ and χ^{-1} are rational functions in λ with simple zeros (poles)

$$\chi^{-1} = \mathbf{c} + \sum_k \frac{\mathbf{A}_k}{\lambda - \mu_k}, \quad \chi = \mathbf{c}^{-1} + \sum_k \frac{\hat{\mathbf{A}}_k}{\lambda - \nu_k}$$

where $\mathbf{c}$, $\mathbf{A}_k$, $\hat{\mathbf{A}}_k$ are matrix functions x and t and sets $\{\mu_k \in \mathbb{C}\}, \{\nu_k \in \mathbb{C}\}$ are constants and

$$\{\mu_k \in \mathbb{C}\} \bigcap \{\nu_k \in \mathbb{C}\} = \emptyset.$$

- How to parametrise matrix valued rational functions?

- What is the dependence of $\mathbf{c}$, $\mathbf{A}_k$, $\hat{\mathbf{A}}_k$ on x and t?

For a scalar function $f(\lambda)$ with simple zeros and poles we need to know: (i) positions of zeros $\{\mu_k \in \mathbb{C}\}$, (ii) positions of poles $\{\nu_k \in \mathbb{C}\}$ and (iii) a value $f(\lambda_0)$ at one regular point λ_0:

$$f(\lambda) = f(\lambda_0) \prod_{k=1}^{N} \frac{(\lambda - \mu_k)(\lambda_0 - \nu_k)}{(\lambda - \nu_k)(\lambda_0 - \mu_k)}.$$

A matrix–valued $(N \times N)$ rational function $\chi(\lambda)$ (with simple poles and zeros) can be uniquely characterised by:

1. a set of zeros $\{\mu_k \in \mathbb{C}\}$ and the corresponding kernel spaces $V_{\mu_k} = \mathrm{Ker}\,\chi(\mu_k)$. The kernel space V_{μ_k} can be seen as a point on the Grassmanian $G_{n_k,N}$, $n_k = \dim V_{\mu_k}$.

2. The set of zeros of the inverse matrix $\{\nu_k \in \mathbb{C}\}$ and corresponding co-kernal spaces $\widehat{V}_{\nu_k} = \mathrm{Ker}\,(\chi^{-1}(\mu_k))^T$

3. A value $\chi(\lambda_0)$ at one regular point λ_0.

Reduction conditions:

$$
\begin{aligned}
&1. \qquad \mathbf{S}\chi(\omega^{-1}\lambda)\mathbf{S}^{-1} = \chi(\lambda)\\
&2. \qquad [\chi^{-1}(\lambda^{-1})]^T = \chi(\lambda)\\
&3. \qquad \chi^*(\lambda^*) = \chi(\lambda)
\end{aligned}
$$

Thus $\nu_k = \mu_k^{-1}$ and set $\{\mu_k\} = \{\mu_k^*\} = \{\omega\mu_k\}$. Therefore the set $\{\mu_k\}$ is a union of orbits of the form:

$$(a) \quad \{\omega^k \mu\}_{k=1}^N, \ \mu \in \mathbb{R}, \qquad \text{"kink", or}$$

$$(b) \quad \{\omega^k \mu, \omega^k \mu^*\}_{k=1}^N, \ \mu \in \mathbb{C}, \ \omega^k \mu \neq \omega^s \mu^* \qquad \text{"breather".}$$

Moreover

$$V_{\omega^k \mu} = \mathbf{S}^k V_\mu, \qquad V_{\mu^*} = V_\mu^*, \qquad \widehat{V}_{\mu^{-1}} = V_\mu.$$

Simplest case, $\mu \in \mathbb{R}$, "kink".

We assume $\mu \neq 0, \pm 1$, $\mu \in \mathbb{R}$:

$$\chi^{-1}(\lambda) = \mathbf{c} + \sum_{k=0}^{N-1} \frac{\omega^{-k}\mathbf{S}^{-k}\mathbf{A}\mathbf{S}^{k}}{\lambda - \omega^{-k}\mu}.$$

Here $\mathbf{c}, \mathbf{A}$ are real (follows from the reduction conditions).

Rank of the kink solution is defined as $\dim V_\mu = \operatorname{rank} \mathbf{A}$.

$$\dim V_\mu = 1 \ \Rightarrow \mathbf{A} = nm^T, \ n \in V_\mu$$

$$\dim V_\mu = k \ \Rightarrow \mathbf{A} = \mathbf{n}\mathbf{m}^T, \ \operatorname{rank} \mathbf{n} = k, \ \operatorname{Span} \mathbf{n} = V_\mu$$

Rank 1 “kink”:

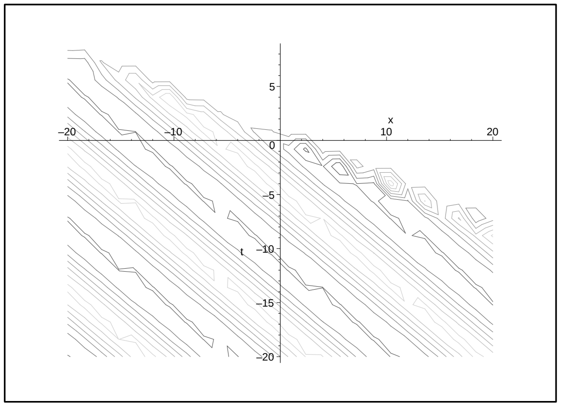
$$\phi_i = \frac{1}{2}\log\left(\frac{\sigma(i-1)\sigma(i+1)}{\sigma(i)^2}\right)$$

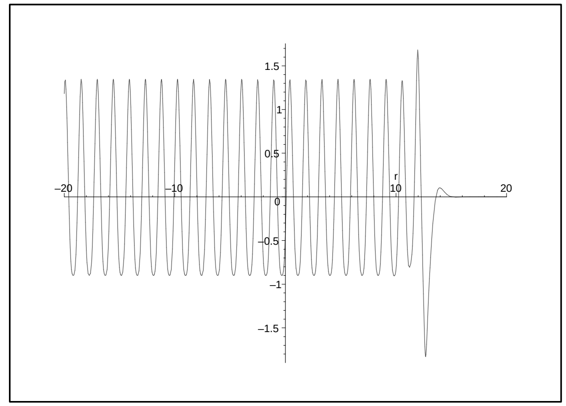
$$\sigma(i) = \sum_{k=1}^{N} n_k^2 \mu^{2\{(i-k) \bmod N\}}$$

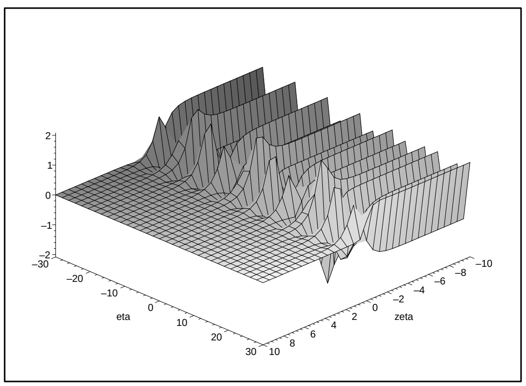
$$\underline{n}(x,t) = \psi_0(\mu,x,t)\underline{n}_0, \qquad \underline{n}_0 \in \mathbb{R}^N$$

$$\psi_0(\lambda,x,t) = e^{(-\lambda^{-1}\Delta+\lambda\Delta^{-1})x+(-\lambda^{-2}\Delta^2+\lambda^2\Delta^{-2})t}$$

$\underline{n}_0 = (1,0,0)$ contour, $t = -5$:



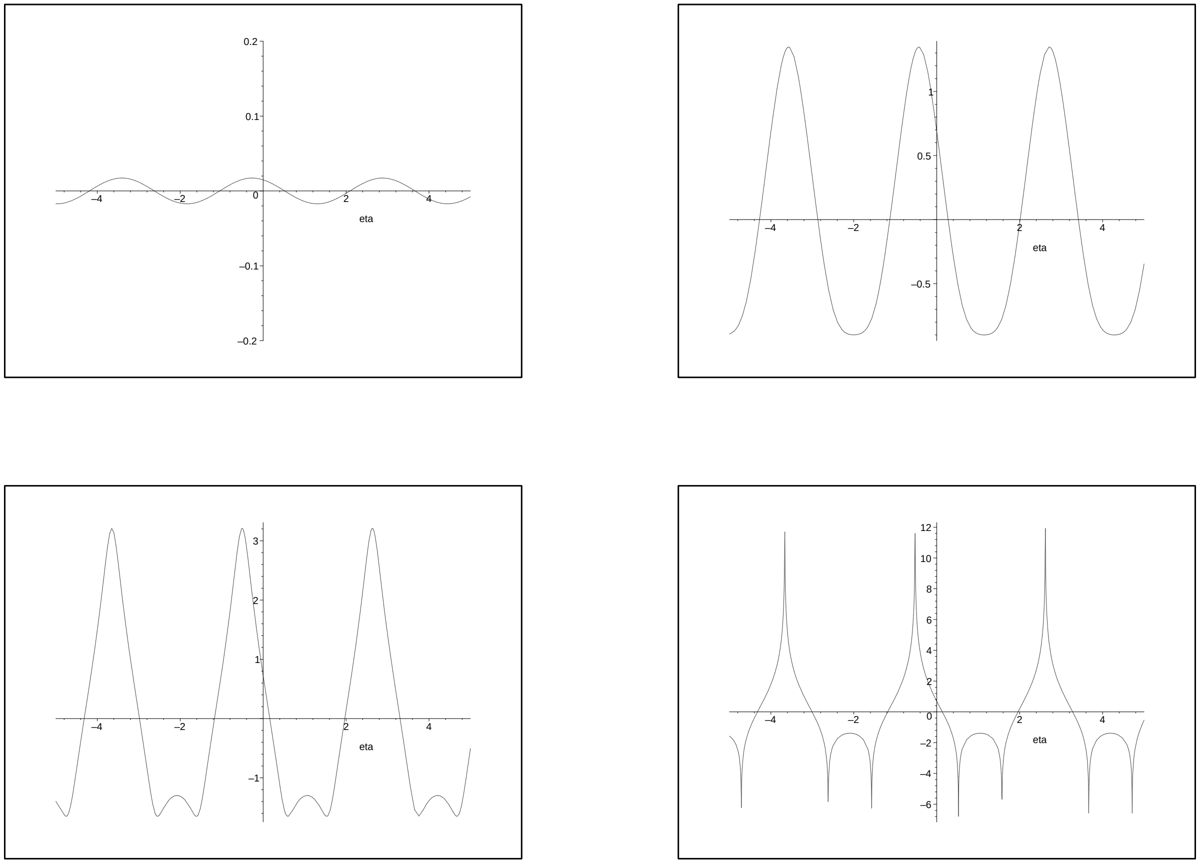




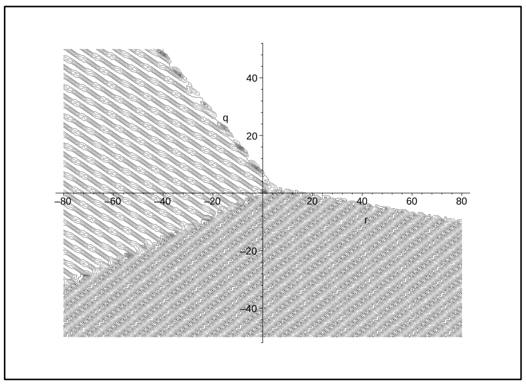
$$\overline{\sigma}(i) = \cos^2(\eta)\mu^{2\{(i-1) \bmod 3\}} + \cos^2(\eta - \frac{4\pi}{3})\mu^{2\{(i-2) \bmod 3\}} + \cos^2(\eta - \frac{2\pi}{3})\mu^{2\{i \bmod 3\}}$$

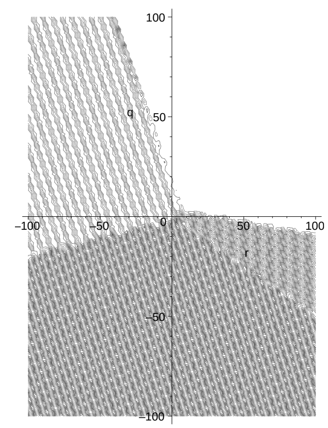
$$\phi_i = \frac{1}{2}\log\left(\frac{\overline{\sigma}(i-1)\overline{\sigma}(i+1)}{\overline{\sigma}(i)^2}\right)$$

$$\mu = 1.01;\ 2;\ 5;\ 1000,\ \mu_c^2 = 1 + \sqrt{3} + \sqrt{3 + 2\sqrt{3}}$$



$N = 5; \ 7$





We can decompose $\mathbb{R}^N$ as a direct sum of invariant subspaces of ψ_0

$$N = 2m-1, \quad \mathbb{R}^N = E_0^1 \bigoplus_{p=1}^{m-1} E_p^2$$

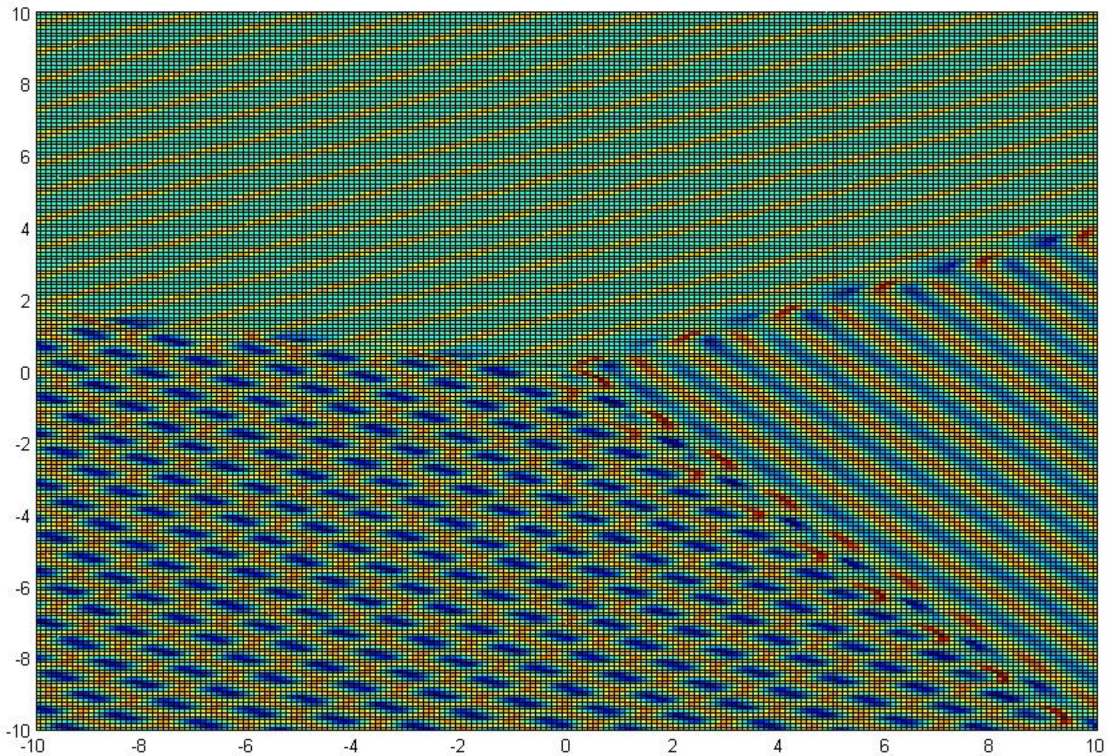
$$N = 2m, \quad \mathbb{R}^N = E_0^1 \bigoplus E_m^1 \bigoplus_{p=1}^{m-1} E_p^2$$

Where

$E_0^1 = \text{span}(\mathbf{e}_N),\ E_m^1 = \text{span}(\mathbf{e}_m),\ E_p^2 = \text{span}(\text{Re}\,(\mathbf{e}_p), \text{Im}\,(\mathbf{e}_p)),$

$$\boldsymbol{\Delta}\mathbf{e}_p = \omega^p \mathbf{e}_p, \quad (\mathbf{e}_p)_k = \omega^{pk} = \exp\left(\frac{2\pi i}{N} pk\right).$$

Rank 2 kink: $\mathbf{n}_0 = ((1,0,0,1,1),(0,1,1,0,1)) \in G_{2,5}$



Complex μ, "breather"

It follows from the reduction group that the simpest form

$$\chi^{-1}(\lambda) = \mathbf{c} + \sum_{k=0}^{N-1} \frac{\omega^{-k}\mathbf{S}^{-k}\mathbf{A}\mathbf{S}^{k}}{\lambda - \omega^{-k}\mu} + \sum_{l=0}^{N-1} \frac{\omega^{-l}\mathbf{S}^{-l}\mathbf{A}^{*}\mathbf{S}^{l}}{\lambda - \omega^{-k}\mu^{*}}$$

We start by assuming that $\mathbf{A}$ is a rank 1 matrix, $\mathbf{A} = \underline{nm}^{T}$.

One can find that

$$c_i^2 = 1 + \frac{N}{\mathbf{D}_i\mathbf{D}_i^* - \mathbf{E}_i\mathbf{E}_i^*}\{\mu^{-1}\mathbf{D}_i^* n_i^2 + (\mu^*)^{-1}\mathbf{D}_i(n_i^*)^2$$

$$-\mu^{-1}\mathbf{E}_i n_i n_i^* - (\mu^*)^{-1}\mathbf{E}_i^* n_i n_i^*\}$$

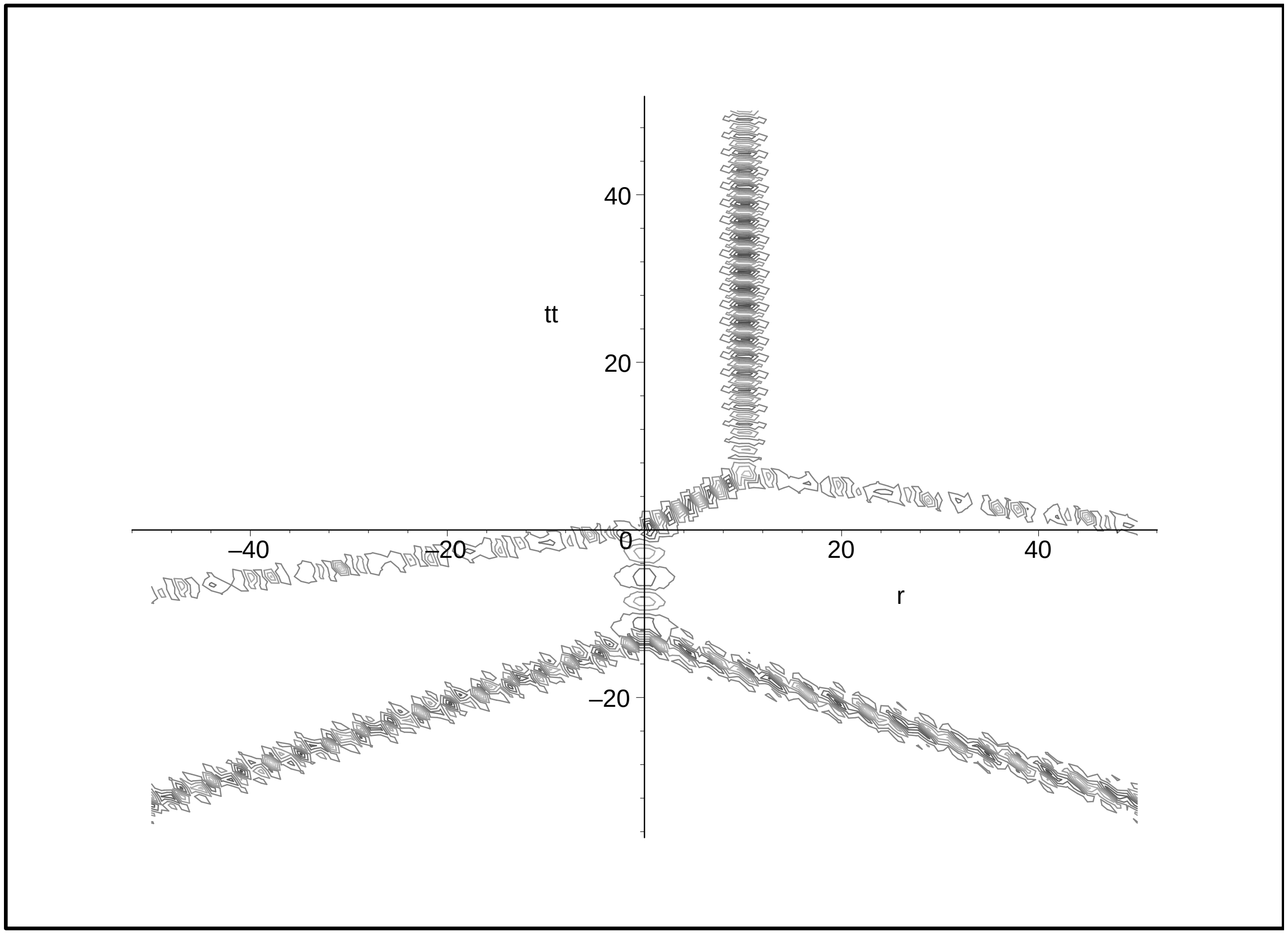
where:

$$\mathbf{D}_i = \frac{N\mu^{-1}}{\mu^{2N} - 1}\sigma(i), \mathbf{E}_i = \frac{N(\mu^*)^{-1}}{|\mu|^{2N} - 1}\rho(i)$$

$$\sigma(i) = \sum_{k=1}^{N} n_k^2 \mu^{2\{(i-k) \bmod N\}}, \rho(i) = \sum_{k=1}^{N} |n_k|^2 |\mu|^{2\{(i-k) \bmod N\}}$$

$$\mathbf{n} = \psi_0(\mu, x, t)\mathbf{n}_0, \quad \mathbf{n}_0 \in \mathbb{C}^N, \qquad \phi_i = \log\left(\frac{c_i}{c_{i+1}}\right).$$

N=5, Rank 1 solution:



There are $N(N-1)/2$ "simple" solitons corresponding 2-dimensional Δ-invariant subspaces of $\mathbb{C}^N$. In the basis $(\mathbf{e}_p)_k = \omega^{pk}$ of eigenvectors $\Delta \mathbf{e}_p = \omega^p \mathbf{e}_p$.

$$\mathbf{n}_0^{pq} = \alpha_p \mathbf{e}_p + \alpha_q \mathbf{e}_q, \qquad \alpha_p, \alpha_q \in \mathbb{C}$$

If $\underline{n}_0 = a_p \mathbf{e}_p + a_q \mathbf{e}_q$ and $\mu = |\mu| e^{i\delta}$ then the width of the soliton is $\frac{1}{\Delta_{pq}}$ where

$$\Delta_{pq} = 2 \left(\frac{1}{|\mu|} - |\mu| \right) \sin \left(\delta - \frac{\pi(p+q)}{N} \right) \sin \left(\frac{\pi(q-p)}{N} \right)$$

its speed is

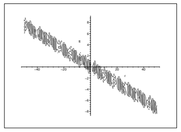
$$v_{pq} = -4 \left(\frac{1}{|\mu|} + |\mu| \right) \cos \left(\delta - \frac{\pi(p+q)}{N} \right) \cos \left(\frac{\pi(q-p)}{N} \right)$$

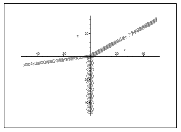
and it is shifted along the x-axis by

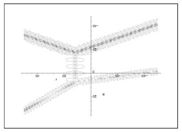
$$x^0_{pq} = \frac{\log \left(\frac{|a_p|}{|a_q|} \right)}{\Delta_{pq}}$$

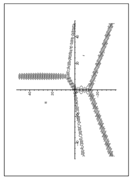
$N = 5$, Classification of soliton configurations. Rank 1:

5 trivial $(10000), (01000), (00100), (00010), (00001)$

10  $(1*000), \ldots, (0010*), \ldots, (0001*)$

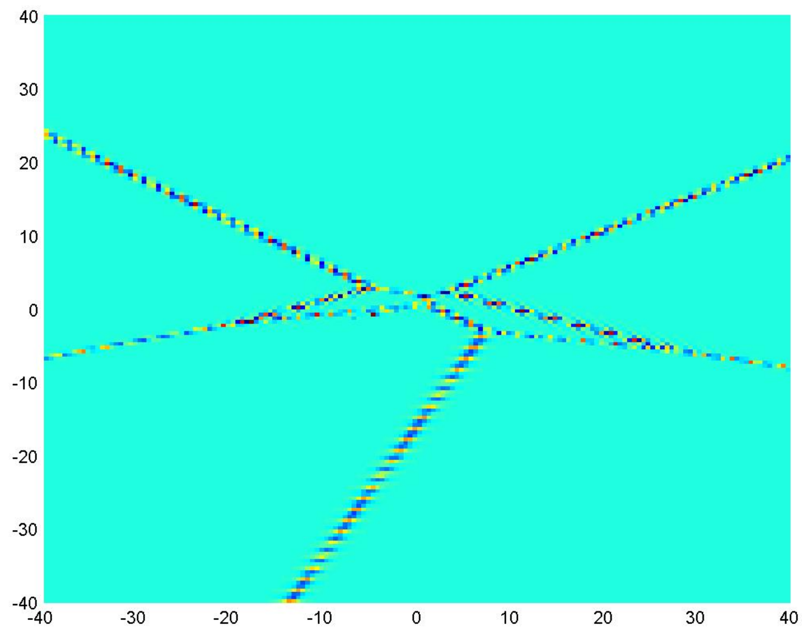
10  $(1**00), \ldots, (010**), \ldots, (001**)$

5  $(1***0), \ldots, (1**0*), \ldots, (01***)$

1 most generic solution  $(1****)$

$N = 5$, Schubert cells and classification of soliton configurations. Rank 2:

The generic cell $\begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{pmatrix}$



1. Continuous limit ($N \to \infty, h = N^{-1}$) to KP, limit of solutions, etc.

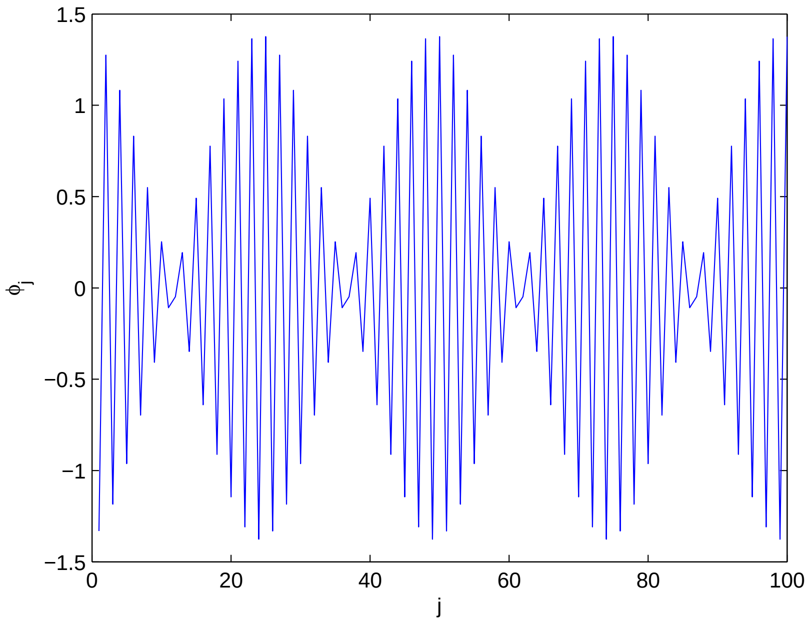
$$\phi_i = \frac{1}{2} \log \frac{\sigma(i+1)\sigma(i-1)}{\sigma(i)^2} \to \frac{h^2}{2} \frac{\partial^2}{\partial \xi^2} \log \sigma(\xi, \eta, \tau)$$

2. "Binary continuous" limit.

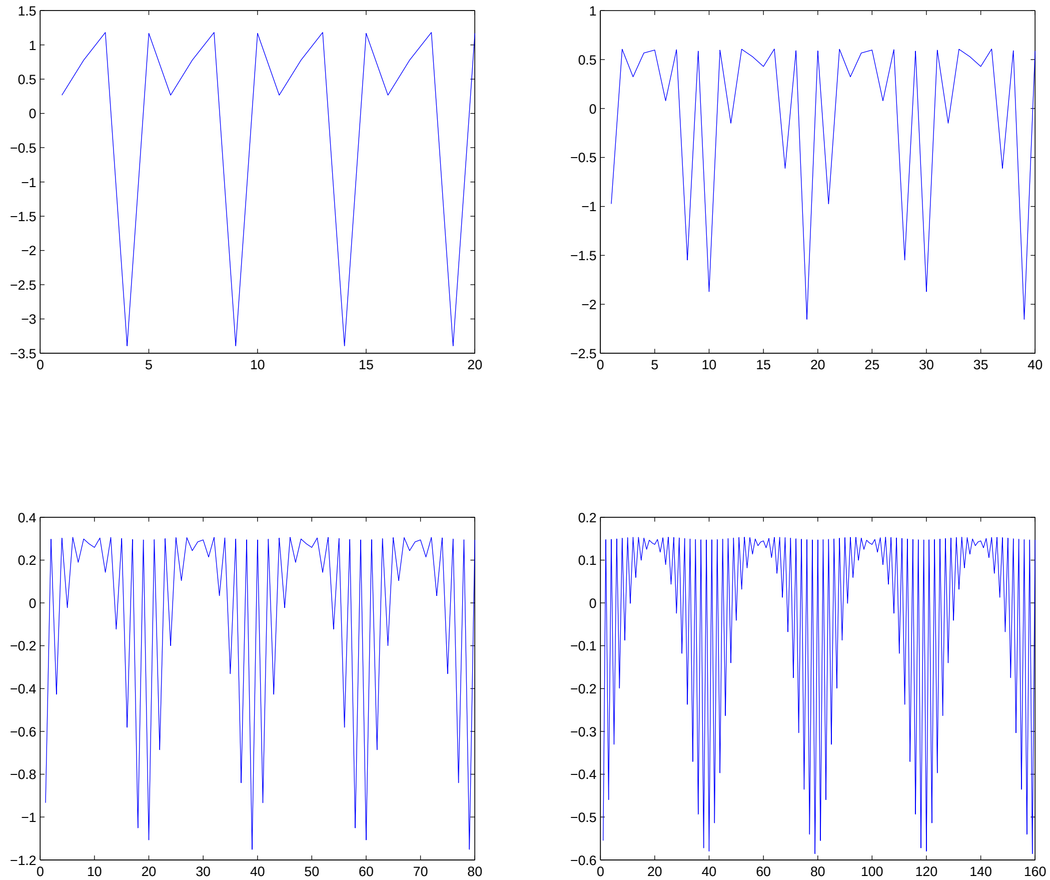
With $p = \frac{N}{4} - 1$, taking the limit as $N \to \infty$ we obtain

$$\phi_j = (-1)^j \log \left(\frac{1 + \mu^2 + (\mu^2 - 1) \cos(2(\mu + \mu^{-1})x + 4\pi y - 2\alpha)}{1 + \mu^2 - (\mu^2 - 1) \cos(2(\mu + \mu^{-1})x + 4\pi y - 2\alpha)} \right)$$

where $y = \frac{j}{N}$.



1.5
1
0.5
0
−0.5
−1
−1.5
φj
0
20
40
60
80
100
j



θ_n, for $N = 20;\ 40;\ 80;\ 160.$