Multidimensional dispersionless integrable hierarchies and their reductions

L.V. Bogdanov

L.D. Landau ITP RAS

General context

- Lax pairs in terms of vector fields (diff. operators of the first order) – Zakharov, Shabat (1979)
- Differential reductions, N-orthogonal coordinate systems – Zakharov (1998). The works of Kyoto school on KP hierarchy reductions (BKP, CKP, etc.)
- Dispersionless limit of integrable systems in (2+1)
- Integrable systems of twistor theory, Plebański heavenly equations and generalizations, hyper-Kähler hierarchies – multidimensional integrable models
- Manakov-Santini hierarchy: generalizes dKP, it is a simplest non-degenerate example of the hierarchy for general vector fields. Dressing method, inverse scattering method for vector fields
- Dunajski interpolating system – describes "a symmetry reduction of the anti-self-dual Einstein equations in (2, 2) signature by a conformal Killing vector whose selfdual derivative is null". On the other hand, it is a simple differential reduction of the Manakov-Santini system

Outline

Recent results on Dunajski-Tod equation and reductions of the generalized dispersionless 2DTL hierarchy, preprint arXiv:1204.3780

1. Introduction
2. The Manakov-Santini system and Dunajski interpolating equation
3. Two-component generalization of the dispersionless 2DTL equation
4. Interpolating differential reductions: elementary description in terms of the Lax pair
5. d2DTL interpolating equation and Dunajski-Tod equation
6. Generalized dispersionless Harry Dym equation and d2DTL interpolating equation
7. Interpolating reductions: general construction
   7.1 Generalized dispersionless 2DTL hierarchy
   7.2 Differential reductions for the hierarchy

Introduction

The equation locally describing general ASD vacuum metric with conformal symmetry (Dunajski and Tod, 1999)

$$(\eta F_{\tilde{w}} + F_{u\tilde{w}})(\eta F_w - F_{uw}) - (\eta^2 F - F_{uu})F_{w\tilde{w}} = 4e^{2\rho u}. \qquad (DT)$$

The equation describing ASD Ricci-flat metric with a conformal Killing vector whose self-dual derivative is null is a simple differential reduction of the Manakov-Santini system (two-component generalization of dKP) (Dunajski 2008).

A scheme of constructing a class of differential reductions for the Manakov-Santini hierarchy (Bogdanov 2008) and general multicomponent case (Bogdanov 2011).

A two-component generalization of the dispersionless 2DTL equation (Bogdanov 2010) and its differential reduction

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty}m_x - m_{xy}m_t). \qquad (*)$$

Equations (DT) and $(*)$ are equivalent up to Legendre transform.

The Manakov-Santini system

The Manakov-Santini system – two-component integrable generalization of the dKP equation,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y. \end{aligned}$$

Lax pair

$$\begin{aligned} &\partial_y \mathbf{\Psi} = ((p - v_x)\partial_x - u_x \partial_p)\mathbf{\Psi}, \\ &\partial_t \mathbf{\Psi} = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p)\mathbf{\Psi}, \end{aligned}$$

where p plays a role of a spectral variable. For $v = 0$ reduces to dKP (Khohlov-Zabolotskaya equation)

$$u_{xt} = u_{yy} + (uu_x)_x,$$

reduction $u = 0$ gives the equation (Pavlov, Martinez Alonso and Shabat)

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y.$$

Dunajski interpolating system

The condition used by Dunajski (JPA 2008) to reduce the Manakov-Santini system to the interpolating system

$$\alpha u = v_x,$$

The reduced MS system can be written as deformed dKP,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_x &= \alpha u, \end{aligned}$$

it also implies a single equation for v,

$$v_{xt} = v_{yy} + \alpha^{-1} v v_{xx} + v_x v_{xy} - v_{xx} v_y.$$

The limit $\alpha \to 0$ corresponds to dKP, $\alpha \to \infty$ – to equation, introduced by Pavlov, Martinez Alonso and Shabat
Dunajski interpolating system describes "a symmetry reduction of the anti-self-dual Einstein equations in (2, 2) signature by a conformal Killing vector whose selfdual derivative is null".

Two-component generalization of d2DTL hierarchy

Two-component generalization of the dispersionless 2DTL equation (L.V. Bogdanov, JPA 43 (2010) 434008)

$$
\begin{aligned}
(\mathrm{e}^{-\phi})_{tt} &= m_t\phi_{xy} - m_x\phi_{ty},\\
m_{tt}\mathrm{e}^{-\phi} &= m_{ty}m_x - m_{xy}m_t.
\end{aligned}
$$

The Lax pair

$$
\begin{aligned}
\partial_x\mathbf{\Psi} &= \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t\frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi},\\
\partial_y\mathbf{\Psi} &= \left(-\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t - \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}
\end{aligned}
$$

For $m = t$ the system reduces to the dispersionless 2DTL equation

$$
(\mathrm{e}^{-\phi})_{tt} = \phi_{xy},
$$

The reduction $\phi = 0$ gives an equation (Pavlov; Shabat and Martinez Alonso)

$$
m_{tt} = m_{ty}m_x - m_{xy}m_t.
$$

Differential reductions in terms of the Lax pair

The Lax pair

$$
\begin{aligned}
\partial_x \Psi &= \hat{U}\Psi, \\
\partial_y \Psi &= \hat{V}\Psi,
\end{aligned}
\tag{1}
$$

where $\hat{U} = u_1\partial_t + u_2\lambda\partial_\lambda$, $\hat{V} = v_1\partial_t + v_2\lambda\partial_\lambda$ are vector fields.
More generally, a one-parametric family of Lax pairs

$$
\begin{aligned}
\partial_x \Phi &= \hat{U}\Phi + \beta\,\mathrm{div}\,\hat{U}\Phi, \\
\partial_y \Phi &= \hat{V}\Phi + \beta\,\mathrm{div}\,\hat{V}\Phi,
\end{aligned}
\tag{2}
$$

where β is a parameter, $\mathrm{div}\,\hat{U} = \partial_t u_1 + \lambda\partial_\lambda u_2$, $\mathrm{div}\,\hat{V} = \partial_t v_1 + \lambda\partial_\lambda v_2$.
In terms of $\ln\Phi$, takes a form of nonhomogeneous linear system

$$
\begin{aligned}
\partial_x \ln\Phi &= \hat{U}\ln\Phi + \beta\,\mathrm{div}\,\hat{U}, \\
\partial_y \ln\Phi &= \hat{V}\ln\Phi + \beta\,\mathrm{div}\,\hat{V},
\end{aligned}
\tag{3}
$$

The Poisson bracket $\{f,g\} = \lambda(f_\lambda g_t - f_t g_\lambda)$ of two solutions of the standard Lax pair $J = \{\Psi_1, \Psi_2\}$, satisfies the system

$$\begin{aligned} \partial_x \ln J &= \hat{U} \ln J + \mathrm{div}\, \hat{U}, \\ \partial_y \ln J &= \hat{V} \ln J + \mathrm{div}\, \hat{V}, \end{aligned} \tag{4}$$

The general solution of the system (1) is of the form $F(\Psi_1, \Psi_2)$, the general solution of nonhomogeneous one-parametric linear system (3) is $\beta\{\Psi_1, \Psi_2\} + f(\Psi_1, \Psi_2)$, and the general solution of the one-paremetric system (2) is $\Phi = \exp(\beta\{\Psi_1, \Psi_2\}) F(\Psi_1, \Psi_2)$. Suggesting the existence of solution f with some special analytic properties in λ for the Lax pairs (2) or (3), we will obtain one-parametric interpolating reduction, which for $\beta = 0$ implies the existence of solution f for standard Lax equations (Gelfand-Dikii type reduction), and in the limit $\beta \to \infty$ corresponds to Hamiltonian (divergence-free) vector fields.

A simplest interpolating reduction

We define a simplest interpolating reduction by the condition that one-parametric Lax equations (2) possess a solution $f = \lambda$ (equivalently, equations (3) possess a solution $\ln\lambda$ and equations (4) – solution $-\alpha\ln\lambda$, $\alpha = -\beta^{-1}$.) An explicit form of equations (3) is

$$\partial_x \mathbf{\Phi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - (\phi_t \frac{m_x}{m_t} - \phi_x)\lambda\partial_\lambda \right) \mathbf{\Phi} + \beta\partial_t \frac{m_x}{m_t},$$
$$\partial_y \mathbf{\Phi} = \left(-\frac{1}{\lambda}\frac{e^{-\phi}}{m_t}\partial_t - \frac{1}{\lambda}\frac{(e^{-\phi})_t}{m_t}\lambda\partial_\lambda \right) \mathbf{\Phi} - \beta\frac{e^{-\phi}}{\lambda}\partial_t \frac{1}{m_t}, \tag{5}$$

the substitution of solution $\ln\lambda$ to both equations gives the same reduction condition

$$\mathrm{e}^{\alpha\phi} = m_t, \quad \alpha = -\beta^{-1},$$

This reduction makes it possible to rewrite the d2DTL system as one equation for m,

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty}m_x - m_{xy}m_t), \tag{$*$}$$

or in the form of deformed d2DTL equation,

$$
\begin{aligned}
&(\mathrm{e}^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{ty}, \\
&m_t = \mathrm{e}^{\alpha\phi}.
\end{aligned}
$$

The limit $\alpha \to 0$ gives the d2DTL equation, the limit $\alpha \to \infty$ gives the equation introduced by Pavlov; Shabat and Martinez Alonso.
Equation $(*)$ is connested with the generalization of a dispersionless $(1+2)$-dimensional Harry Dym equation, Blaszak (2002), and also with an equation describing ASD vacuum metric with conformal symmetry, Dunajski and Tod (1999) (see below)

Legendre transform and Dunajski-Tod equation

Equation $(*)$ can be represented in exterior differential form

$$\beta^{-1}\mathrm{d}m_t^{\beta} \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}m_y \wedge \mathrm{d}m \wedge \mathrm{d}y, \qquad (6)$$

where $\beta = 1 - \alpha^{-1}$.

Let us consider a Legendre type transform (where τ is a new independent variable and M is a new dependent variable)

$$m_t = e^{\tau}, \quad M = m - te^{\tau}.$$

Differential of M is of the form

$$\mathrm{d}M = M_x \mathrm{d}x + M_y \mathrm{d}y - te^{\tau}\mathrm{d}\tau.$$

Transformed equation (6) reads

$$\beta^{-1}\mathrm{d}e^{\beta\tau} \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}M_y \wedge \mathrm{d}M \wedge \mathrm{d}y - \mathrm{d}M_y \wedge \mathrm{d}M_\tau \wedge \mathrm{d}y,$$

and transformed equation $(*)$ looks like

$$e^{\beta\tau} = (M_{y\tau}M_x - M_{yx}M_\tau) - (M_{y\tau}M_{x\tau} - M_{yx}M_{\tau\tau}). \qquad (7)$$

Scaling the time $\tau \to 2\tau$, we get

$$4e^{2\beta\tau} = 2(M_{y\tau}M_x - M_{yx}M_\tau) - (M_{y\tau}M_{x\tau} - M_{yx}M_{\tau\tau})$$

In terms of the function $F = e^{-\tau}M$

$$(F_y + F_{y\tau})(F_x - F_{x\tau}) - (F - F_{\tau\tau})F_{xy} = 4e^{-2\alpha^{-1}\tau}.$$

Considering the scaling $x \to \eta^{-1}x$, $y \to \eta^{-1}y$, $\tau \to \eta\tau$, we obtain Dunajski-Tod equation

$$(\eta F_y + F_{y\tau})(\eta F_x - F_{x\tau}) - (\eta^2 F - F_{\tau\tau})F_{xy} = 4e^{2\rho\tau}, \qquad (DT)$$

where $\rho = -\alpha^{-1}\eta$.

Generalized dispersionless Harry Dym equation and d2DTL interpolating equation

Generalized dispersionless Harry Dym equation constructed by Blaszak (2002) can be written in the form of conservation law,

$$\partial_t u^{2-r} = \frac{(3-r)}{(r-1)(r-2)} \left(u^{2-r}\partial_x^{-1}\partial_y u^{r-1}\right)_y, \tag{8}$$

r is integer, $r \in \mathbb{Z}$. This equation suggests the existence of potential v, such that

$$\begin{aligned} \partial_y v &= u^{2-r}, \\ \partial_t v &= \frac{(3-r)}{(r-1)(r-2)} u^{2-r}\partial_x^{-1}\partial_y u^{r-1}, \end{aligned}$$

and for the potential we get an equation

$$\frac{(3-r)}{(r-1)(r-2)} v_{yy} = v_y^{\frac{r-3}{2-r}}(v_{xt}v_y - v_t v_{xy}),$$

which after the change of variables $y \longrightarrow t$, $x \longrightarrow y$, $t \longrightarrow x$ is equivalent to equation $(*)$ with $\alpha = \frac{2-r}{r-3}$.

Interpolating reductions: general construction

The generalized dispersionless 2DTL hierarchy

We consider formal series

$$\Lambda^{\text{out}} = \ln\lambda + \sum_{k=1}^{\infty} l_k^{+}\lambda^{-k}, \quad \Lambda^{\text{in}} = \ln\lambda + \phi + \sum_{k=1}^{\infty} l_k^{-}\lambda^{k},$$

$$M^{\text{out}} = M_0^{\text{out}} + \sum_{k=1}^{\infty} m_k^{+}\mathrm{e}^{-k\Lambda^{\text{out}}}, \quad M^{\text{in}} = M_0^{\text{in}} + m_0 + \sum_{k=1}^{\infty} m_k^{-}\mathrm{e}^{k\Lambda^{\text{in}}},$$

$$M_0 = t + \sum_{k=1}^{\infty} x_k \mathrm{e}^{k\Lambda} - \sum_{k=1}^{\infty} y_k \mathrm{e}^{-k\Lambda}, \tag{9}$$

where λ is a spectral variable, t, x_k, y_k are considered independent variables, and other coefficients of the series (ϕ, m_0, $l_k^{\pm}$, $m_k^{\pm}$) – dependent variables. Usually we suggest that 'out' and 'in' components of the series define the functions outside and inside the unit circle in the complex plane of the variable λ respectively.

Generalized dispersionless 2DTL hierarchy is defined by the generating relation

$$(\{\Lambda, M\}^{-1}\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{out}} = (\{\Lambda, M\}^{-1}\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{in}}. \tag{10}$$

The differential d is given by

$$\mathrm{d}f = \partial_\lambda f \mathrm{d}\lambda + \partial_t f \mathrm{d}t + \sum_{k=1}^{\infty} \partial_k^+ f \mathrm{d}x_k + \sum_{k=1}^{\infty} \partial_k^- f \mathrm{d}y_k, \tag{11}$$

where $\partial_k^+ f = \frac{\partial f}{\partial x_k}$, $\partial_k^- f = \frac{\partial f}{\partial y_k}$. As a result of condition (10), the coefficients of the differential two-form in the generating relation (10) are *meromorphic*. Generating equation (10) implies Lax-Sato equations of the hierarchy:

$$\left(\partial_n^+ - \left(\frac{\mathrm{e}^{n\Lambda}\lambda\partial_\lambda\Lambda}{\{\Lambda, M\}}\right)_+^{\text{out}} \partial_t + \left(\frac{\mathrm{e}^{n\Lambda}\partial_t\Lambda}{\{\Lambda, M\}}\right)_+^{\text{out}} \lambda\partial_\lambda\right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \tag{12}$$

$$\left(\partial_n^- + \left(\frac{\mathrm{e}^{-n\Lambda}\lambda\partial_\lambda\Lambda}{\{\Lambda, M\}}\right)_-^{\text{in}} \partial_t - \left(\frac{\mathrm{e}^{-n\Lambda}\partial_t\Lambda}{\{\Lambda, M\}}\right)_-^{\text{in}} \lambda\partial_\lambda\right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \tag{13}$$

where $(\dots)_-$, $(\dots)_+$ are standard projections respectively to negative and nonnegative powers of λ.

Interpolating reduction for the hierarchy

Rewriting Lax-Sato equations symbolically as

$$(\partial_n^+ - \hat{U}_n)\begin{pmatrix}\Lambda\\ M\end{pmatrix} = 0, \qquad (\partial_n^- - \hat{V}_n)\begin{pmatrix}\Lambda\\ M\end{pmatrix} = 0, \tag{14}$$

Non-homogeneous linear equations for the Jacobian $J_0 = \{\Lambda, M\}$

$$\begin{aligned}\partial_n^+ \ln J_0 &= \hat{U}_n \ln J_0 + \operatorname{div} \hat{U}_n,\\ \partial_n^- \ln J_0 &= \hat{V}_n \ln J_0 + \operatorname{div} \hat{V}_n.\end{aligned} \tag{15}$$

We define interpolating reduction for the hierarchy by the condition

$$(\ln J_0 - \alpha\Lambda)^{\text{out}} = (\ln J_0 - \alpha\Lambda)^{\text{in}} \tag{16}$$

This relation implies that

$$(\ln J_0 - \alpha\Lambda) = -\alpha \ln \lambda, \tag{17}$$

thus nonhomogeneous linear equations of the hierarchy possess a solution $f = -\alpha \ln \lambda$.

Substituting the expression for the Poisson bracket implied by relation (17),

$$J_0 = \{\Lambda, M\} = \lambda^{-\alpha} \exp(\alpha\Lambda),$$

to the generating relation (10), we obtain the generating relation for the reduced hierarchy

$$(\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{out}} = (\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{in}}.$$

The Lax-Sato equations for the reduced hierarchy read

$$\left(\partial_n^+ - \left(\lambda^\alpha \mathrm{e}^{(n-\alpha)\Lambda}\lambda\partial_\lambda\Lambda\right)_+^{\text{out}} \partial_t + \left(\lambda^\alpha \mathrm{e}^{(n-\alpha)\Lambda}\partial_t\Lambda\right)_+^{\text{out}} \lambda\partial_\lambda\right) \begin{pmatrix}\Lambda \\ M\end{pmatrix} = 0,$$
$$\left(\partial_n^- + \left(\lambda^\alpha \mathrm{e}^{(-n-\alpha)\Lambda}\lambda\partial_\lambda\Lambda\right)_-^{\text{in}} \partial_t - \left(\lambda^\alpha \mathrm{e}^{(-n-\alpha)\Lambda}\partial_t\Lambda\right)_-^{\text{in}} \lambda\partial_\lambda\right) \begin{pmatrix}\Lambda \\ M\end{pmatrix} = 0.$$

Similar to d2DTL hierarchy, Lax-Sato equations for Λ split out, having no interaction with M.

The characterization of reductions in terms of the dressing data

A dressing scheme for the two-component 2DTL hierarchy

$$\begin{aligned}\Lambda^{\text{in}} &= F_1(\Lambda^{\text{out}}, M^{\text{out}}),\\ M^{\text{in}} &= F_2(\Lambda^{\text{out}}, M^{\text{out}}),\end{aligned}$$

$\Lambda^{\text{in}}(\lambda, \mathbf{t})$, $M^{\text{in}}(\lambda, \mathbf{t})$ are analytic inside the unit circle with punctured zero, the functions $\Lambda^{\text{out}}(\lambda, \mathbf{t})$, $M^{\text{out}}(\lambda, \mathbf{t})$ are analytic outside the unit circle with prescribed singularities defined by the series.

The Riemann problem implies that the differential form

$$\Omega_0 = \frac{\mathrm{d}\Lambda \wedge \mathrm{d}M}{\{\Lambda, M\}}$$

is meromorphic and the generating relation for the hierarchy.

To provide the generating relation for the reduced hierarchy

$$(\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{out}} = (\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{in}},$$

the Riemann-Hilbert problem should be area-preserving in tefms of the variables

$$f_1(\Lambda, M) = \Lambda, \quad f_2(\Lambda, M) = e^{-\alpha\Lambda}M,$$

or, symbolically,

$$(\Lambda^{\text{in}}, M^{\text{in}}) = \mathbf{F}(\Lambda^{\text{out}}, M^{\text{out}}),$$

the reduction condition for the dressing data reads

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \mathrm{SDiff}(2).$$

(a twisted area-preservation condition). Another choice of **f** leads to higher reductions.

Higher reductions

We define higher interpolating reductions by the condition

$$(\ln J_0 - aL^n - bL^{-n})^{\text{out}} = (\ln J_0 - aL^n - bL^{-n})^{\text{in}},$$

where $L = e^\Lambda$.
Reduction condition (18) implies the expression for the Poisson bracket,

$$J_0 = \{\Lambda, M\} = \exp(aL^n + bL^{-n} - aL_+^n - bL_-^{-n}), \tag{18}$$

which is valid for both 'in' and 'out' components. The generating relation for the reduced hierarchy

$$(\exp(-aL^n - bL^{-n})\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{out}} = (\exp(-aL^n - bL^{-n})\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{in}}.$$

Lax-Sato equations for Λ split out, similar to d2DTL hierarchy.

A case $n = 1$

Let us consider a case $n = 1$ in more detail. Nonhomogeneous Lax-Sato equations (15) in this case possess a rational solution

$$f = -aL_+ - bL_-^{-1} = -a(\lambda + l_1^+) - b\frac{e^{-\phi}}{\lambda}$$

The simplest form of the differential reduction for the d2DTL system reads

$$m_{tt} = a(\phi_t m_x - \phi_x m_t) - b(m_t)^2\phi_y.$$

However, in this form the differential relation contains all the variables x, y, t. It is possible rewrite it in equivalent form in (x, t) plane or (y, t) plane. The differential reduction in (y, t) plane reads

$$\partial_y\partial_t \ln m_t = -a\partial_t\left(\frac{(e^{-\phi})_t}{m_t}\right) - b\partial_y(m_t\phi_y).$$

Considering this reduction together with interpolating reduction $\mathrm{e}^{\alpha\phi} = m_t$, we obtain a (1+1)-dimensional system which represents a reduction of equation $(*)$ and can be rewritten as (1+1)-dimensional equation for the function m

$$a m_{tt} = (m_t)^{\frac{1}{\alpha}+1}(\alpha m_{ty} + b m_{yy} m_t). \tag{19}$$

It is possible to transform this equation to the system of hydrodynamic type.

The differential reduction in (x, t) plane reads

$$\partial_t \left(a \frac{m_x}{m_t}(\phi_t \frac{m_x}{m_t} - \phi_x) + \partial_t \frac{m_x}{m_t} + b(e^{-\phi})_t \right) - a\partial_x \left(\phi_t \frac{m_x}{m_t} - \phi_x \right) = 0.$$

Together with interpolating reduction, it forms a (1+1)-dimensional system representing a reduction of equation $(*)$,

$$b m_{tt} m_t^{-\frac{1}{\alpha}-1} - \alpha\partial_t \frac{m_x}{m_t} + a \left(\frac{m_x}{m_t}\partial_t - \partial_x \right) \frac{m_x}{m_t} = 0. \tag{20}$$

A common solution of (1+1)-dimensional equations (19), (20) gives a solution of equation $(*)$.

THANK YOU!