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Solitons, Collapses and Self-Similar Solutions in Cahn-Hilliard Kind Equation

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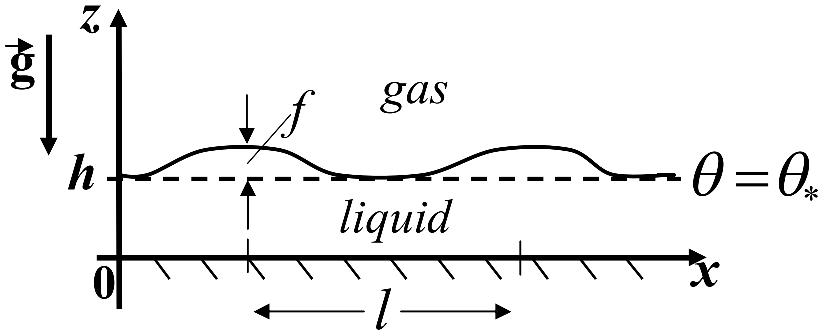
Statement of the problem



$$f\,l h^{-2} \to const \quad as \quad h l^{-1} \to 0$$

- A two-dimensional a thin horizontal layer of a viscous fluid with thermal inhomogeneity in the presence of gravity force is considered.
- Liquid layer bounded by a planar solid substrate from below and by a free surface from above.
- The bottom temperature θ_0 is assumed to be constant and more that the gas temperature θ_1.
- The interfacial tension liquid-gas interface $\sigma = \sigma_0 + \phi(\theta - \theta_*)^2$

  σ_0, ϕ, θ_* are positive constants.
- The characteristic disturbance amplitude of free surface *u(x,y,t)* is much less than the average layer thickness h.

In the thin layer approximation the Rayleigh-Benard problem with the condition $\theta = \theta_*$ on the free surface is investigated.

Cauchy problem

The evolution of the non-dimensional deviation of a free boundary from a horizontal equilibrium state *u(x,y,t)* can be described in terms of Cauchy problem solutions for the equation of Cahn-Hilliard kind

$$u_t + \Delta^2 u + \Delta(u^2 - \beta u) = 0; \quad u = u_0(x, y), \ t = 0 \qquad (1)$$

$\beta = \rho g h^2 / \sigma_0$ is the Bond number

$u_0(x, y)$

- double periodic function, or
- rapidly decreasing function at $x, y \to \infty$

"Mass" conservation law $$\iint_{R^2} u(x, y, t)\,dxdy = \iint_{R^2} u_0(x, y, t)\,dxdy = c \qquad (2)$$

Global existence of periodic solution

$\Pi = \{x, y : 0 < x < 2\pi, 0 < y < 2\pi\kappa^{-1}\}, \kappa \geq 1$

$\varepsilon = \varepsilon(\beta, \kappa)$ is sufficient small positive number

Let $u_0 \in H_0^2(\Pi)$ and $\| u_0 \|_{H_0^2} \leq \varepsilon$, where $H_0^2(\Pi)$ is the subspace of Sobolev space formed by periodic function.

If $\beta > -1$, Cauchy problem (1) has a unique generalized solution

$$u(x, y, t) \in L^2(0, \infty; H_0^2(\Pi))$$

There exist constants $\gamma \in (0, 1+\beta)$ and $C > 0$ independante of t such that the estimate $e^{\gamma t} \| u_0 \|_{L^2} \leq C\varepsilon$ is true for any fixed $t > 0$.

The condition of smallness $\| u_0 \|_{H_0^2}$ is essential for the global existence of solution of problem (1). Solution having a "large" initial norm can be destroyed for a finite time.

Space-periodic solutions of Cauchy problem and rapidly decreasing solutions at infinity are studied.

Self-similar solutions of axially symmetric problem

$$u = t^{-1/2} f(\xi), \quad \xi = t^{-1/4}(x^2 + y^2)^{1/2} \quad (\beta = 0) \quad (3)$$

$$[\xi^{-1}(\xi f')']' - \frac{1}{4}\xi f + 2ff' = 0,$$

$$|f| < \infty, \quad \xi \to 0; \quad f \to 0, \quad \xi \to \infty$$

Singular points: regular point $\xi = 0$ and irregular point $\xi = \infty$.

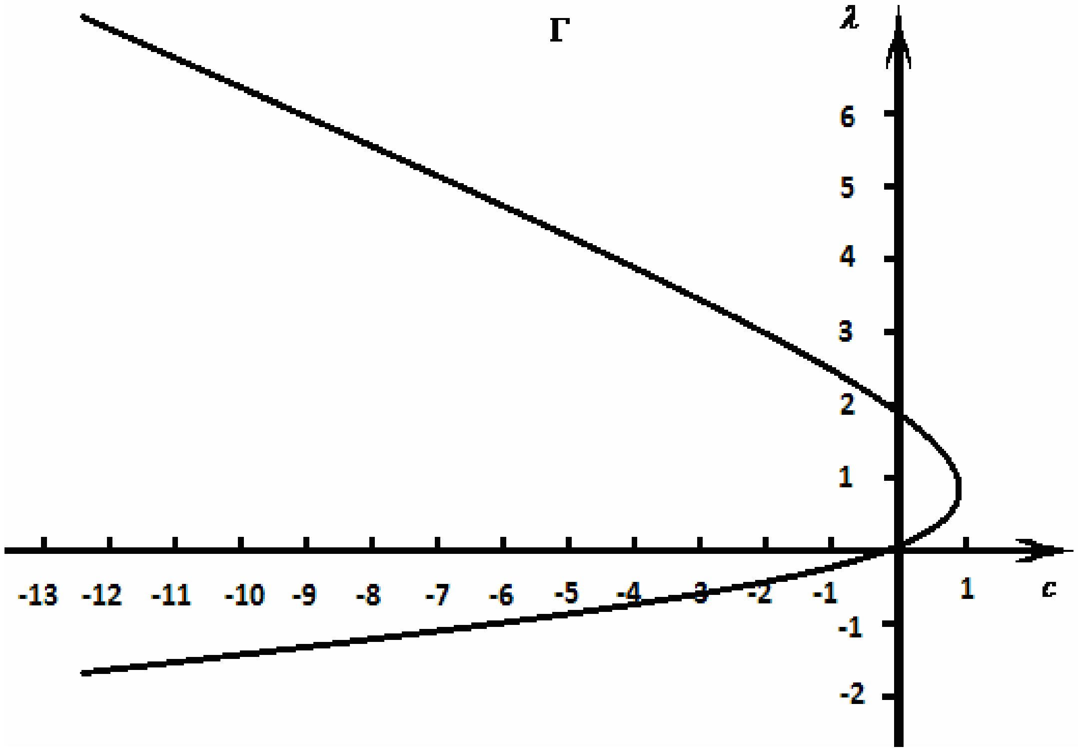
We seek non-trivial solutions that are defined for all $\xi > 0$, regular at $\xi \to 0$, and rapidly decreasing at $\xi \to \infty$. Such solutions form one-parameter family with the parameter c, where

$$c = \int_0^\infty \xi f d\xi$$

Axially symmetric self-similar solutions

exist at small values of $|c|$,

do not exist for large and positive c, $c \le c_* \simeq 0.8155$.



$$2f''(0) + f^2(0) = -\frac{c}{4}$$

Fig. 1. Curve Γ is a double-valued function $\lambda = f(0)$ of the parameter c.

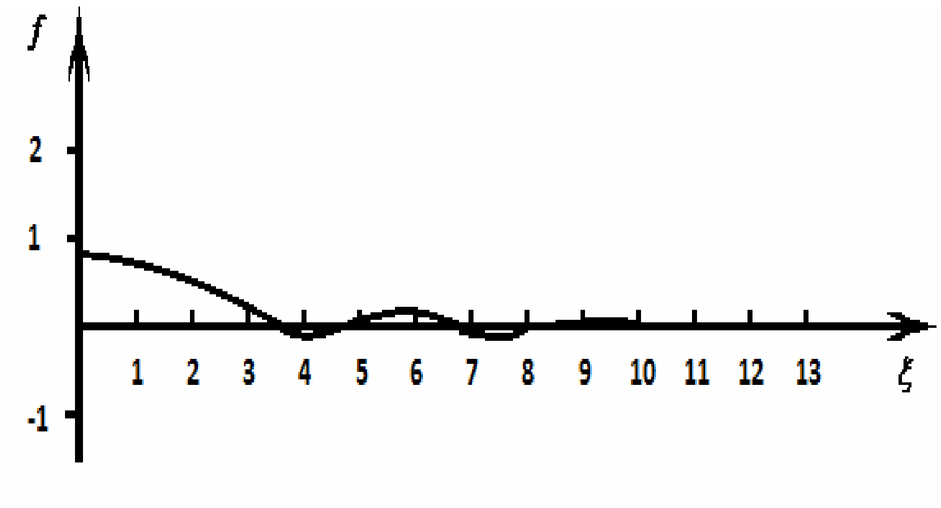


Fig. 2. $c = c_* \simeq 0.8155$; $\lambda = 0.860$

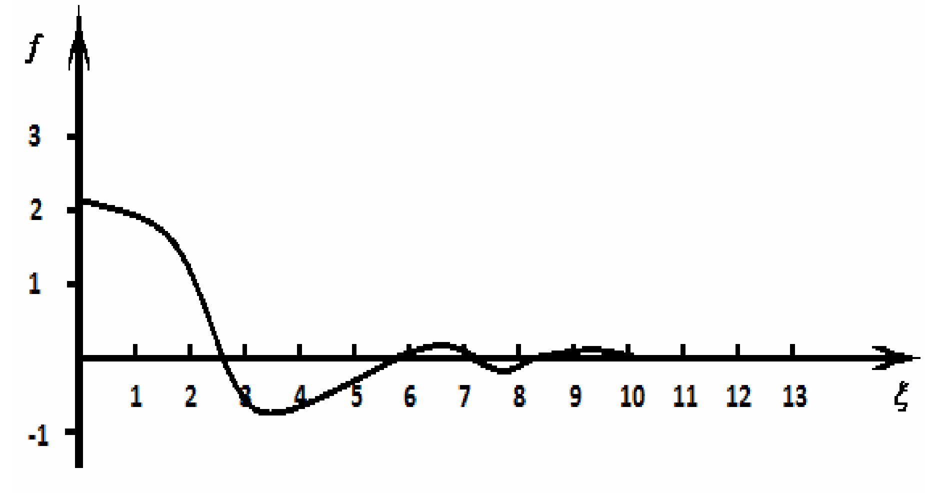


Fig. 3. $c = 0$; $\lambda = 2.057$

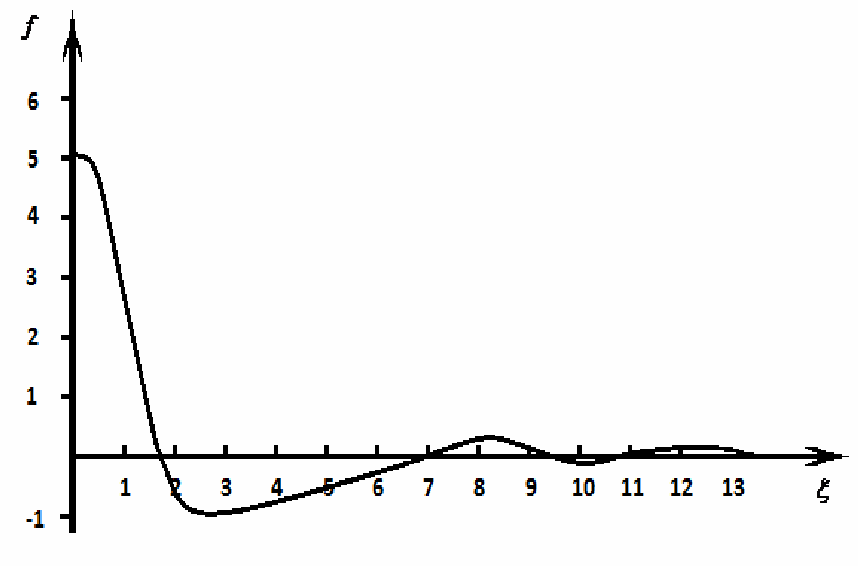


Fig. 4. $c = -6$; $\lambda = 5.019$

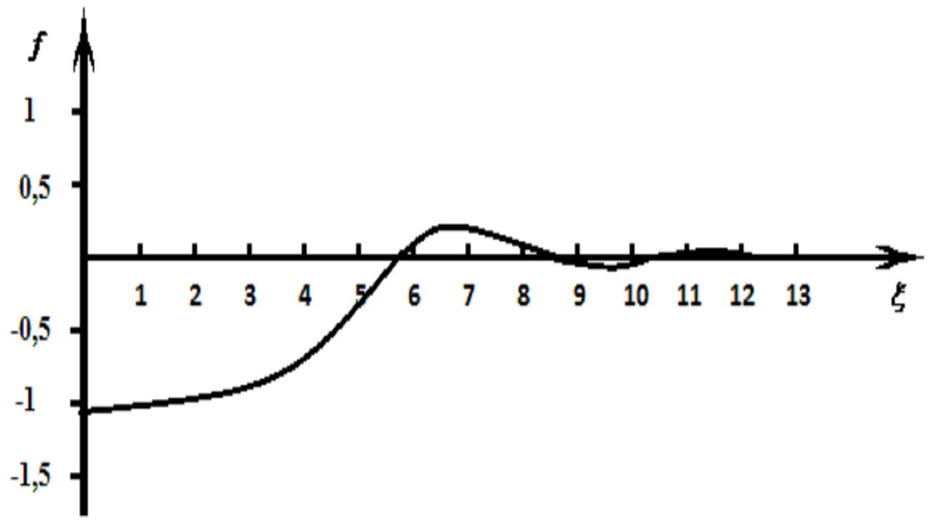


Fig. 5. $c = -6$; $\lambda = -1.122$

Self-similar solutions of plane problem

$$u = t^{-1/2}\varphi(\eta), \quad \eta = xt^{-1/4} \tag{4}$$

$$\varphi'''' + (\varphi^2)'' - \frac{\eta}{4}\varphi' - \frac{1}{2}\varphi = 0,$$

$$\varphi'(0) = 0;$$

$$\varphi \to 0, \quad \eta \to \infty.$$

$$\int_{-\infty}^{\infty} \varphi \, d\eta = 0$$

$\varphi(\eta)$ is even function

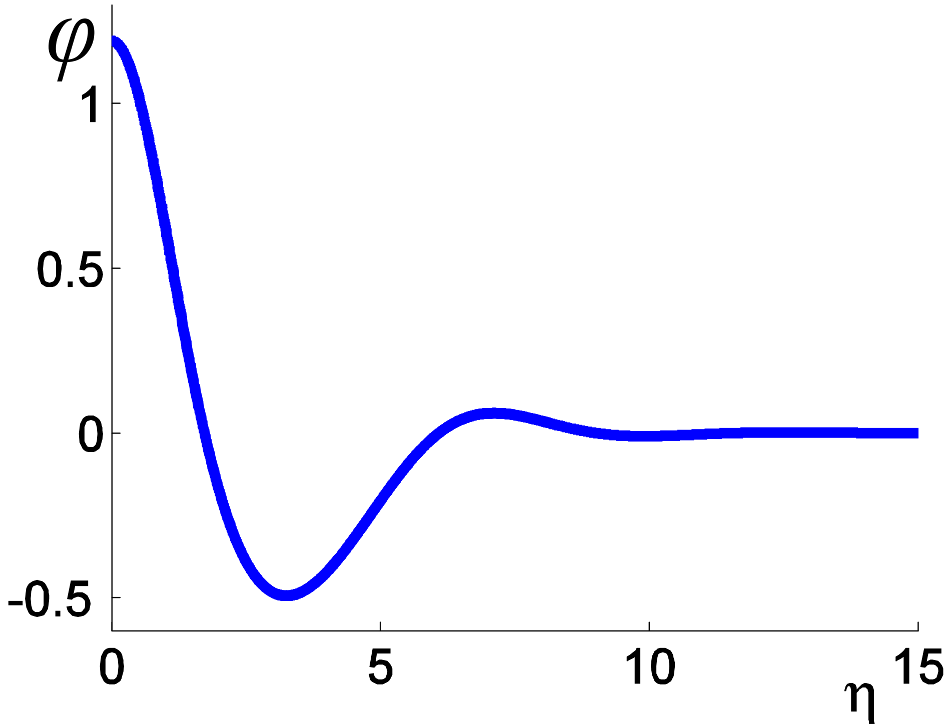


Fig. 6.

Lyapunov functional

$$\frac{dS(u)}{dt} = \iint_{\Pi} |\nabla[\Delta u + (u - \beta/2)^2]|^2 \, dx\, dy$$

$$S(u) = \iint_{\Pi} \left(\frac{1}{3}(u - \beta/2)^3 - \frac{1}{2}|\nabla u|^2 \right) dx\, dy \qquad (5)$$

First variation $\delta S = \iint_{\Pi} (\Delta u + u^2 - \beta u)\delta u \, dx\, dy$

Gradient form $u_t = \mathrm{grad}_{H^{-2}} S(u)$

Each **stationary** solution u_s of equation (1) is the extremal point for the functional *S(u)*. Critical points of S are saddle points as a rule.

Second variation $\delta^2 S(u_s) = \iint_{\Pi} (-|\nabla \delta u|^2 + (2u_s - \beta)(\delta u)^2) dx\, dy$

Stationary solutions

- cnoidal waves,
- Korteweg and de Vries solitons, $u_s = 3\beta/(2\cosh^2(x\sqrt{\beta}/2))$
- axially symmetric solitons, $u_s = g(\sqrt{x^2+y^2})$
- travelling waves do not exist, $u = q(x-ct)$

Sufficient condition for stability of the stationary solution u_s

$$2u_s < 1 + \beta \qquad (6)$$

Stationary solution of Eq. (1) may be found as a solution of the evolutionary problem.

Evolutionary problem
2π-periodic initial function

$$u_0(x) = a_1 \cos x + a_2 \cos 2x \qquad (7)$$

$$u(x,t) \approx u^{(N)}(x,t) = \sum_{n=1}^{N} u_n(t)\cos(nx), \quad u_n(0) = a_n, \quad n = 1,\dots,N \qquad (8)$$

$$a_1 = a_2 = 1, \quad a_i = 0, i = 3,\dots,N$$

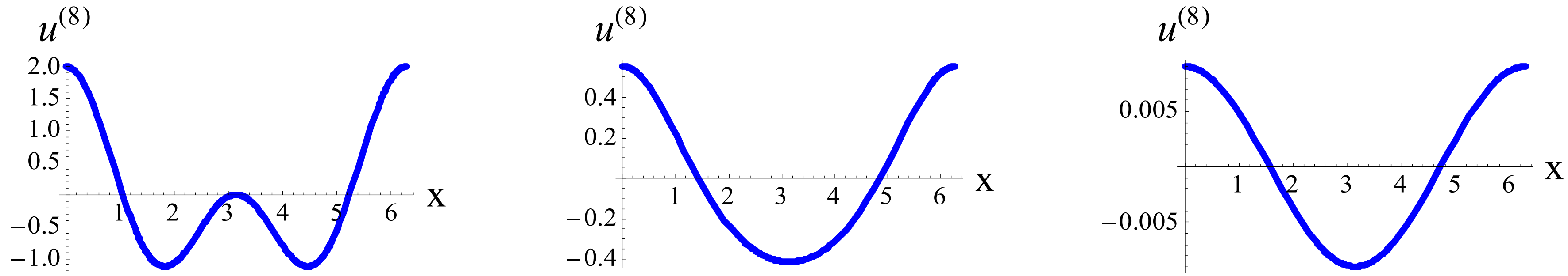


Fig. 7. (a) t=0, (b) t=1, (c) t=5.

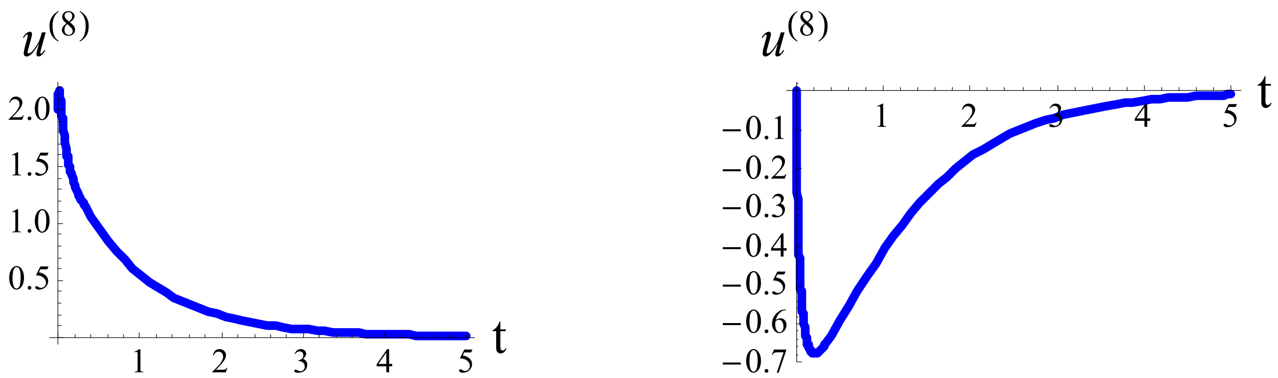


Fig. 8. (a) x=0, x=2π, (b) x=π.

Evolutionary problem non-periodic initial function

$$u(x,t) \approx u^{(N)}(x,t) = \sum_{n=0}^{N} v_n(t) C_n(x), \quad v_n(0) = q_n, \quad n = 0,\ldots,N \qquad (9)$$

Christov's functions

$$C_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1} (-1)^{n+k+1} \binom{2n+1}{2k-2} x^{2k-2}}{(x^2+1)^{n+1}}, \quad n = 0,1,2,\ldots \qquad (10)$$

Initial data $\quad q_0 = 1,\ q_1 = -2,\ q_2 = 1, \quad q_i = 0,\ i = 3,\ldots,N$

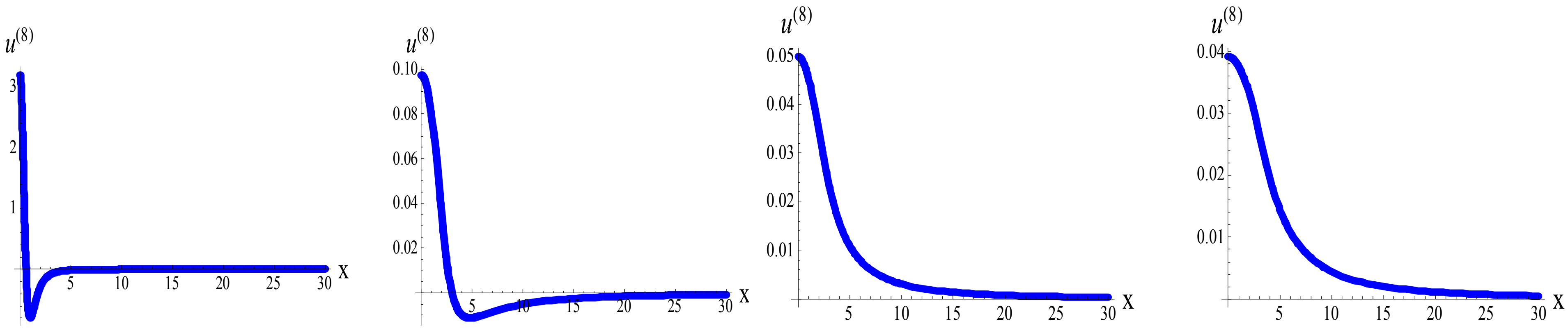


Fig. 9. (a) t = 0, (b) t = 5, (c) t = 10, (d) t = 20.

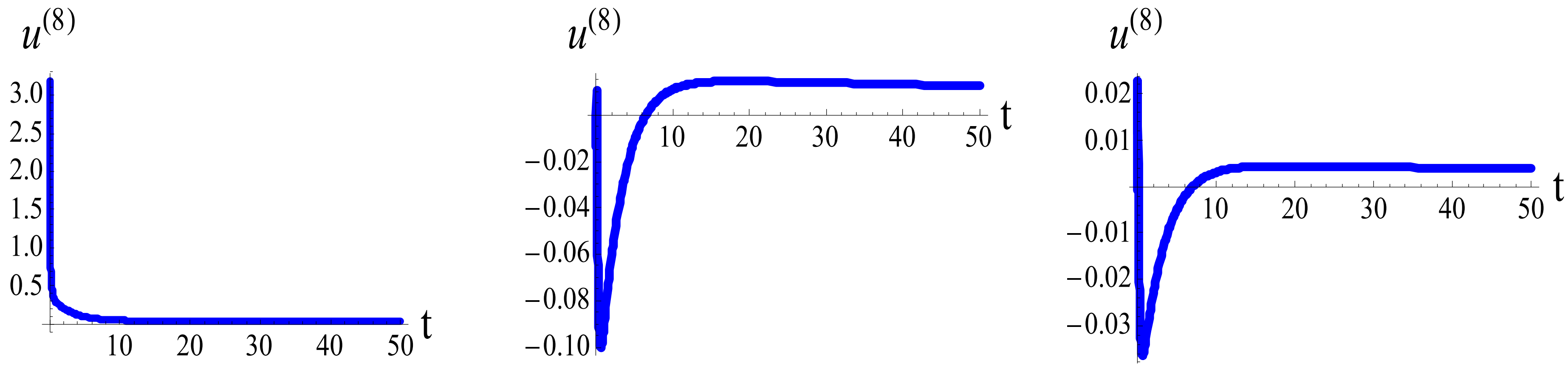


Fig. 10. (a) x = 0, (b) x = 5, (c) x = 10.

Collapsing solutions

The behavior of Cauchy problem solutions (1) are following:

- $u \to u_s$ when $t \to \infty$, where u_s is some stationary solution, or
- its solution is destroyed for a finite or infinite time.

Sufficient condition of collapse existence

Proposition. *Let initial function* $u_0 \in H_0^2(\Pi)$ *satisfied the inequality*

$$\iint_{\Pi} \left(\frac{u_0^3}{3} - \frac{|\nabla u_0|^2}{2} \right) dx\, dy > \frac{6}{5}(1+\beta^2) \iint_{\Pi} \left((-\Delta)^{-1/2} u_0 \right)^2 dx\, dy. \qquad (11)$$

There exist such $t_* > 0$ *that for solution* u *of Cauchy problem (1) we have*

$$\| (-\Delta)^{-1/2} u \|_{L^2} \to \infty \quad when \quad t \to t_* - 0.$$

The inequality (11) can not be fulfilled for "small" data u_0, and also for odd function u_0.

Solutions having a "large" initial norm can be destroyed for a finite time.

Simple example (β = 0)

$$u_0(x) = a_1 \cos x + a_2 \cos 2x,$$

$$|a_1| > 2, \quad a_1^2 - \sqrt{a_1^4 - 16} < 2a_2 < a_1^2 + \sqrt{a_1^4 - 16}.$$

$$a_1 = 2.1, \quad a_2 = 2$$

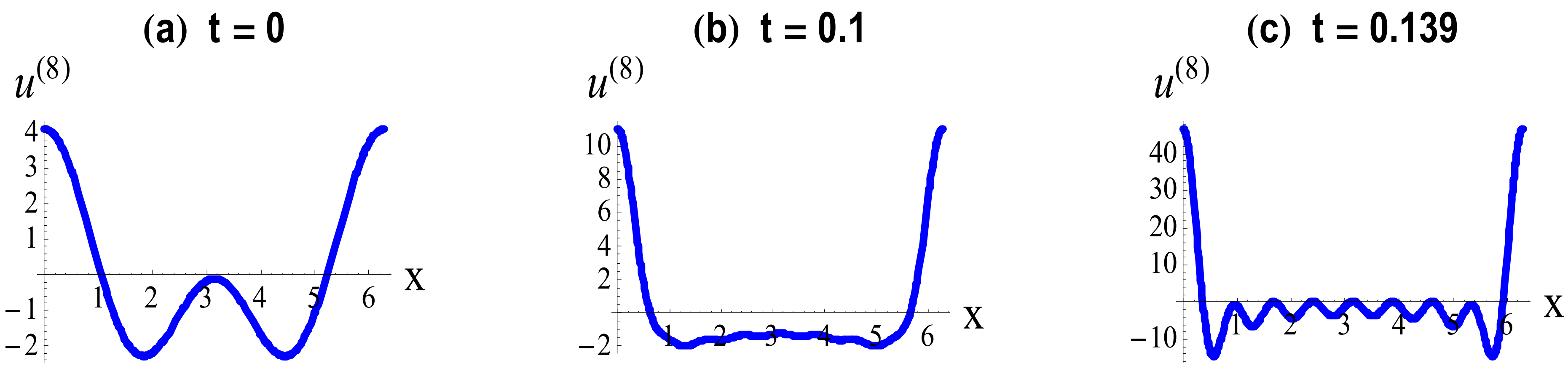


Fig. 11.

Numerical calculations show that the solution of a non-periodic problem collapses for finite time.

$$q_0 = 1, q_1 = 1, q_2 = -7/5, q_3 = 3/7, \quad q_i = 0, i = 4, \ldots, N$$

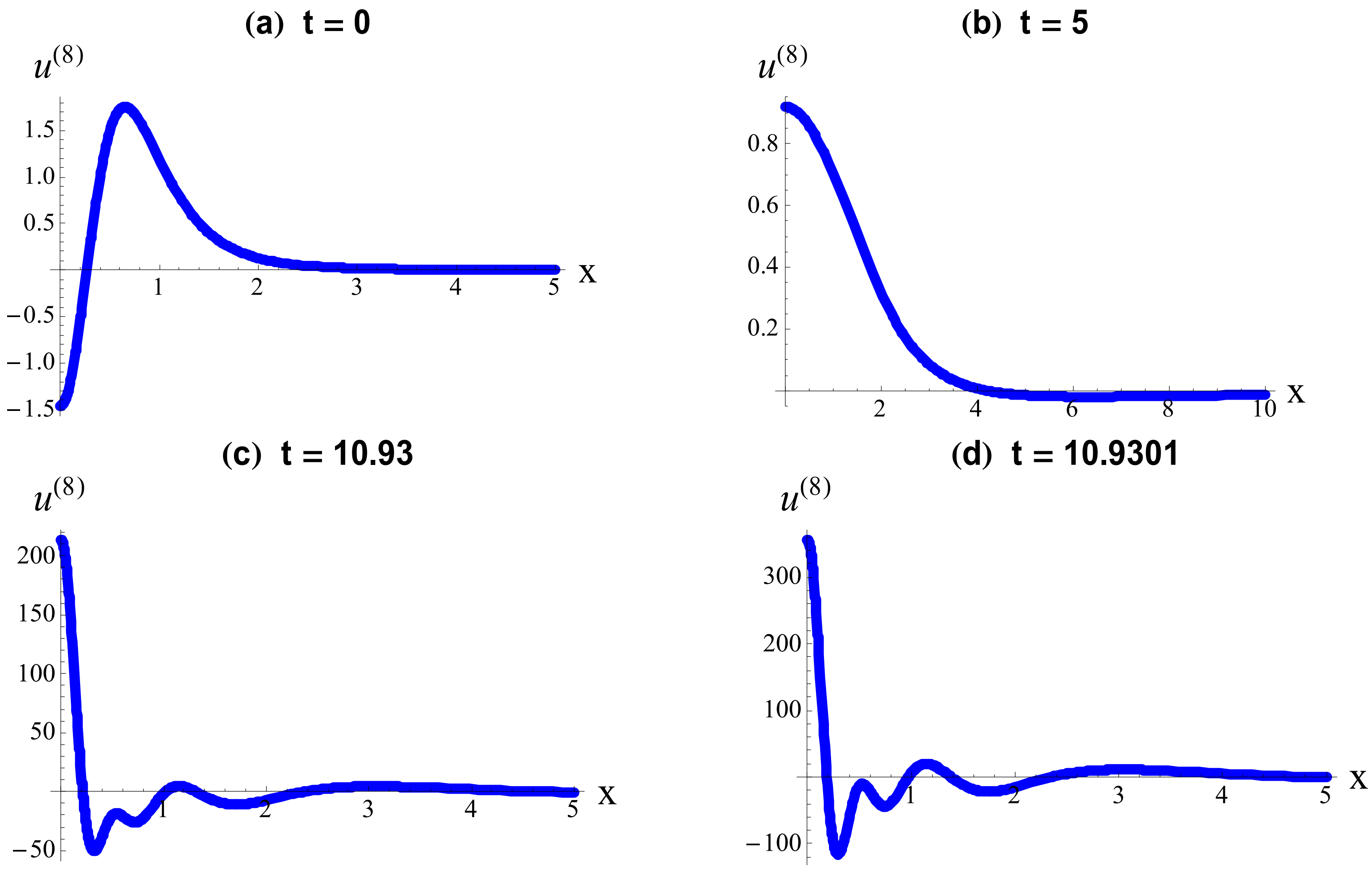


Fig. 12.

Conclusions

- The sufficient instability condition of the equilibrium has been obtained in the framework of the long-wave approximation.
- The sufficient condition of the global solution existence of problem (1) and its collapse for a finite time for the periodic initial function has been formulated.
- Analytical and numerical research shows that axially symmetric self-similar solutions exist at small values of $|c|$, where c is a constant in mass conservation law (2), and they do not exist for large and positive c. For negative values of c there were found two branches of self-similar solutions with various qualitative behaviors. Such solutions form one-parameter family with the parameter c.
- The self-similar solutions of the plane problem satisfying the conservation law exist only for $c = 0$.
- Korteweg and de Vries solitons, axially symmetric solitons, cnoidal waves are stationary solutions of the problem.
- There are no nontrivial stationary solutions in the form of travelling waves.

Thank you for your attention!