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Solvability of the initial boundary value problem for the equations of viscous compressible multifluids

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Barotropic compressible NS ($n > 1$), global results

Theory of global weak solutions for $p = \rho^\gamma$, γ large enough. Here we should distinguish between stationary and non-stationary problems.

- Kazhikhov, Weigant (1995): global existence, smoothness ($n = 2$).
- P.-L. Lions (1993, 1998, 1999): global weak solvability of basic boundary value problems ($p = \rho^\gamma$, $\gamma > 3$): communicative relations for effective viscous flux

$$\overline{\rho \cdot (p - \nu \mathrm{div}\, \boldsymbol{u})} = \overline{\rho} \cdot \overline{(p - \nu \mathrm{div}\, \boldsymbol{u})}.$$

- Further progress: Feireisl, Matusu-Necasova, Petzeltova, Straskraba, Novotny (1998, 1999, 2001): $\gamma \searrow 3/2, \dots$
- Feireisl (2004–2007): $\lim$ via Ma, Fr ...; analysis near the boundary; heat-conductive models.
- P.L.Lions (1998), Feireisl (2004, 2009), Novotny, Straskraba (2004): review of compressible NS.

Now the threshold value is $\gamma = 3/2$.

Barotropic compressible NS ($n > 1$), global results

Stationary problems:

- P.L.Lions (1993–1999), Feireisl (2004), Novo, Novotny, Straskraba (2002–2006), Plotnikov, Sokolowski (2004–2012), Frehse, Goj, Steinhauer (2005), Jessle, Novotny (2013), Plotnikov, Weigant (2015): global existence.

Now the threshold value is $\gamma = 1$.
Open problems: regularity, uniqueness, . . .

Mathematical model of multifluids

Models consists in continuity equation for each constituent

$$\frac{\partial \rho_i}{\partial t} + \mathrm{div}(\rho_i \boldsymbol{u}_i) = 0, \quad i = 1, \ldots, N,$$

momentum equation for each constituent

$$\frac{\partial(\rho_i \boldsymbol{u}_i)}{\partial t} + \mathrm{div}(\rho_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla p_i - \mathrm{div}\, \mathbb{S}_i = \rho_i \boldsymbol{f}_i + \boldsymbol{J}_i,\ i = 1, \ldots, N,$$

and the energy equations.

Here ρ_i are the densities, $\boldsymbol{u}_i$ are the velocities p_i are the pressures, $\mathbb{S}_i$ are the viscous parts of the stress tensors $\mathbb{P}_i = -p_i \mathbb{I} + \mathbb{S}_i$ of each constituent:

$$\mathbb{S}_i = \sum_{j=1}^{N} \left(\lambda_{ij} \mathrm{div}\, \boldsymbol{u}_j \mathbb{I} + 2\mu_{ij} \mathbb{D}(\boldsymbol{u}_j) \right), \quad i = 1, \ldots, N,$$

where λ_{ij}, μ_{ij} are viscosity coefficients, $\mathbb{D}$ means the rate of deformation (strain) tensor, and $\mathbb{I}$ is the identity tensor.

Mathematical model of multifluids

Hence, the viscosities λ_{ij}, μ_{ij} and $\nu_{ij} = \lambda_{ij} + 2\mu_{ij}$ ("total" viscosities) form the matrices

$$\boldsymbol{\Lambda} = \{\lambda_{ij}\}_{i,j=1}^{N}, \quad \mathbf{M} = \{\mu_{ij}\}_{i,j=1}^{N}, \quad \mathbf{N} = \{\nu_{ij}\}_{i,j=1}^{N}.$$

$\boldsymbol{f}_i = (f_{i1}, \dots, f_{iN})$ are external forces, and

$$\boldsymbol{J}_i = \sum_{j=1}^{N} a_{ij}(\boldsymbol{u}_j - \boldsymbol{u}_i), \quad i = 1, \dots, N, \quad a_{ij} = a_{ji}, \quad i, j = 1, \dots, N$$

stands for the momentum supply for each constituent.

That means: beside external mass forces $\boldsymbol{f}$, there exist internal mass forces between the constituents, and internal surface forces arise not only inside each constituent but also between them.

Diagonal entries of viscosity matrices are responsible for internal friction inside each constituent, and non-diagonal entries are responsible for friction between constituents.

Mathematical model of multifluids

Properties of the viscosity matrices:

- Onsager principle implies that viscosity matrices must be symmetric, but it causes strong mathematical difficulties (see below).
- Anyway, it is very important (physically and mathematically) to validate "Second law of thermodynamics" which means

$$\sum_{i=1}^{N} \mathbb{S}_i : \mathbb{D}(\boldsymbol{u}_i) \geqslant 0. \tag{1}$$

Moreover, in order to provide ellipticity we claim

$$\sum_{i=1}^{N} \int_{\Omega} \mathbb{S}_i : \mathbb{D}(\boldsymbol{u}_i) d\boldsymbol{x} \geqslant C \sum_{i=1}^{N} \int_{\Omega} |\nabla \otimes \boldsymbol{u}_i|^2 d\boldsymbol{x}, \tag{2}$$

where Ω is the flow domain, and $\boldsymbol{u}_i|_{\partial\Omega} = 0$.

Mathematical model of multifluids

The formulated positiveness or ellipticity can be provided by the following properties of viscosity matrices:

$$n\mathbf{\Lambda} + 2\mathbf{M} \geqslant 0, \qquad \mathbf{M} \geqslant 0 \tag{3}$$

provide (1),

$$\mathbf{\Lambda} + 2\mathbf{M} > 0, \qquad \mathbf{M} > 0 \tag{4}$$

provide (2), etc. (n is the dimension of the flow).

Important: viscosity matrices are not diagonal!

Momentum supply $\boldsymbol{J}_i$ gives lower order terms (physically important, but mathematically causing no difficulties), and if the matrices are diagonal then $\boldsymbol{J}_i$ is the only connection between two constituents, so we have two NS systems connected only via l.o.t.
Earlier such problems were relevant (even in 1D), but now such results almost automatically come from the theory of one fluid (usual compressible NS).
If viscosity matrices are "complete" then we have interesting mathematical problems.

Mathematical results for homogeneous multifluids

Up to now: only approximate models.

- Frehse, Goj, Malek (2002, 2005): stationary Stokes system without convective terms (solvability in 3D space, uniqueness under additional restrictions).
- Frehse, Weigant (2007): quasi-stationary model (3D, bounded domain, special boundary conditions, classic solutions).
- Kucher, Prokudin (2009): stationary model (barotropic case, bounded domain 3D, triangle matrix of total viscosities).
- Kucher, Mamontov, Prokudin (2012), Mamontov, Prokudin (2014): steady heat-conductive (with one temperature of multi-temperature) models (bounded domain 3D, triangle matrix of total viscosities).

Main problem

Automatic extension of the theory of compressible NS to multifluids requires

$$\operatorname{div}\operatorname{div}\mathbb{S}_i = \mathrm{const}_i \cdot \Delta \operatorname{div} \boldsymbol{u}_i, \quad i = 1, \dots, N,$$

but we have

$$\begin{bmatrix} \operatorname{div}\operatorname{div}\mathbb{S}_1 \\ \dots \\ \operatorname{div}\operatorname{div}\mathbb{S}_N \end{bmatrix} = \mathbf{N} \begin{bmatrix} \Delta \operatorname{div} \boldsymbol{u}_1 \\ \dots \\ \Delta \operatorname{div} \boldsymbol{u}_N \end{bmatrix}.$$

It is possible to obtain results for triangle matrix **N**. However, for general matrix **N**, the method is to be developed.

We first succeeded to consider the case of general total viscosity matrices, i. e., to escape any restrictions on their structure, except some natural properties (such as positive definiteness) which are related with fundamental physical laws.

Assumptions

- Pressures in the constituents are equal.
- Material derivative in the constituents is based on the average velocity of the multifluid.

Meanwhile,

- Both assumptions are physically reasonable in some situations.
- The mathematical model does not loose the variety of multifluid models (different densities and velocities are preserved), and moreover, it unfolds completely because
- The assumptions listed above allow to remove restrictions on the viscosity matrix and to take into account all summands in the viscosity terms.

Denote

$$\boldsymbol{v} = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{u}_i, \qquad \rho = \sum_{i=1}^{N} \rho_i \tag{1}$$

are the average velocity and complete density of the multifluid, respectively.

Assumptions

Note that

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v}) = \operatorname{div}(\rho_i(\boldsymbol{v} - \boldsymbol{u}_i)), \quad i = 1, \ldots, N, \tag{2}$$

$$\frac{\partial(\rho_i \boldsymbol{u}_i)}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) + \underline{\operatorname{div}(\rho_i(\boldsymbol{u}_i - \boldsymbol{v}) \otimes \boldsymbol{u}_i) - \boldsymbol{J}_i} + \\ + \nabla p_i = \operatorname{div} \mathbb{S}_i + \rho_i \boldsymbol{f}_i, \;\; i = 1, \ldots, N. \tag{3}$$

The underlined summands in the momentum equations (3), and the right-hand sides of the continuity equations (2) are small under the assumption that the phase velocities $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ of the constituents are close to each other. This assumption is justified physically due to the equalizing of the velocities which takes place via the collisions of the molecules in homogeneous mixtures.

Let us also suppose that in all constituents the pressures are equal $p_1 = \ldots = p_N = p$ and are defined by the total density ρ.

Assumptions

Thus, we come to the following equations:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v}) = 0, \quad i = 1, \ldots, N, \tag{4}$$

$$\frac{\partial(\rho_i \boldsymbol{u}_i)}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) + \nabla p(\rho) = \operatorname{div} \mathbb{S}_i + \rho_i \boldsymbol{f}_i, \quad i = 1, \ldots, N \tag{5}$$

for N scalar and N vector-valued (total $4N$ scalar) unknown functions, where the relation between p and ρ (i. e. the function $p(\cdot)$) is given. Note that the momentum equations (7) may be rewritten as

$$\rho_i \frac{\partial \boldsymbol{u}_i}{\partial t} + \rho_i (\nabla \otimes \boldsymbol{u}_i)^* \boldsymbol{v} + \nabla p(\rho) = \operatorname{div} \mathbb{S}_i + \rho_i \boldsymbol{f}_i, \quad i = 1, \ldots, N,$$

and $(\nabla \otimes \boldsymbol{u}_i)^* \boldsymbol{v} = (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}_i$. This (non-divergent) form is inconvenient for weak solutions, but it allows to see (common for all equations) operator of the material derivative $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$.

Statement of the problem

Problem A.

In the closure $\overline{Q}_T$ of the domain $Q_T = (0, T) \times \Omega$, where $\Omega \subset \mathbb{R}^3$ is the flow domain, $T > 0$ is an arbitrary real number, it is required to define the scalar fields of the densities $\rho_i \geqslant 0$, $i = 1, \ldots, N$, and the vector fields of the velocities $\boldsymbol{u}_i$, $i = 1, \ldots, N$, which satisfy the following system of equations and initial/boundary conditions:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v}) = 0, \quad i = 1, \ldots, N, \tag{6}$$

$$\frac{\partial(\rho_i \boldsymbol{u}_i)}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) + \nabla p(\rho) = \operatorname{div} \mathbb{S}_i + \rho_i \boldsymbol{f}_i, \quad i = 1, \ldots, N \tag{7}$$

$$\rho_i|_{t=0} = \rho_{0i}, \qquad \rho_i \boldsymbol{u}_i|_{t=0} = \boldsymbol{q}_i, \quad i = 1, \ldots, N, \tag{8}$$

$$\boldsymbol{u}_i|_{(0,T)\times\partial\Omega} = 0, \quad i = 1, \ldots, N. \tag{9}$$

Statement of the problem

Here the pressure p is defined by the total density ρ via the polytropic equation of state

$$p(\rho) = K\rho^{\gamma} \tag{10}$$

with some constants $K > 0$ and $\gamma > 3/2$.

The initial data in the Problem A will be taken in the class

$$\begin{gathered}\rho_{0i} \in L_{\gamma}(\Omega), \quad \rho_{0i} \geqslant 0, \quad \mathrm{mes}(\{\rho_{0i} = 0\} \bigcap \{\boldsymbol{q}_i \neq 0\}) = 0, \\ \frac{|\boldsymbol{q}_i|^2}{\rho_{0i}} \in L_1(\Omega), \quad i = 1, \ldots, N.\end{gathered} \tag{11}$$

Statement of the problem

Consider the values (initial velocities)

$$\boldsymbol{u}_{0i}(\boldsymbol{x}) = \begin{cases} \dfrac{\boldsymbol{q}_i(\boldsymbol{x})}{\rho_{0i}(\boldsymbol{x})}, & \rho_{0i}(\boldsymbol{x}) > 0, \\ \\ \text{extended arbitrarily}, & \rho_{0i}(\boldsymbol{x}) = 0. \end{cases}$$

Suppose that

$$\boldsymbol{u}_{0i} \in L_{\frac{2\gamma}{\gamma-1}}(\Omega), \quad i = 1, \ldots, N, \tag{12}$$

then $\boldsymbol{u}_{0i} \cdot \boldsymbol{q}_i \in L_1(\Omega)$, $i = 1, \ldots, N$.

Finally, the external body forces, for the simplicity, are supposed to satisfy

$$\boldsymbol{f}_i \in L_\infty(Q_T), \quad i = 1, \ldots, N. \tag{13}$$

Statement of the problem

Definition. Weak solution to the Problem A is called the collection of functions

$$\rho_i \in L_\infty(0,T;L_\gamma(\Omega)), \quad \rho_i \geqslant 0, \quad \boldsymbol{u}_i \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \quad i=1,\ldots,N,$$

which satisfy the following conditions:

(1) The densities ρ_i satisfy the continuity equations (6) and the initial conditions (8) in the sense that for all $\phi_i \in C_0^1\big([0,T);C^\infty(\overline{\Omega})\big)$ the following integral identities take place

$$\int\limits_{Q_T} \Big(\rho_i \frac{\partial \phi_i}{\partial t} + \rho_i \boldsymbol{v}\cdot\nabla\phi_i\Big)\, d\boldsymbol{x}dt + \int\limits_{\Omega} \rho_{0i}\phi_i|_{t=0}\, d\boldsymbol{x} = 0, \ \ i=1,\ldots,N;$$

Statement of the problem

(2) The velocities $\boldsymbol{u}_i$ satisfy the momentum equations (7) and the initial conditions (8) in the sense that for all vector fields $\boldsymbol{\varphi}_i \in C_0^1([0,T); C_0^\infty(\Omega))$ the following integral identities take place

$$\int\limits_{Q_T} \Big(\rho_i \boldsymbol{u}_i \cdot \frac{\partial \boldsymbol{\varphi}_i}{\partial t} + (\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) : (\nabla \otimes \boldsymbol{\varphi}_i) + p(\rho) \operatorname{div} \boldsymbol{\varphi}_i + \rho_i \boldsymbol{f}_i \cdot \boldsymbol{\varphi}_i \Big) \, d\boldsymbol{x} \, dt =$$

$$= \int\limits_{Q_T} \mathbb{S}_i : (\nabla \otimes \boldsymbol{\varphi}_i) \, d\boldsymbol{x} \, dt - \int\limits_{\Omega} \boldsymbol{q}_i \cdot \boldsymbol{\varphi}_i(0, \boldsymbol{x}) \, d\boldsymbol{x}, \quad i = 1, \dots, N$$

(the boundary conditions (9) are valid automatically in the sense of the class $\overset{\circ}{W}{}_2^1(\Omega)$).

The main result

Theorem. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of the class $C^{2+\nu_1}$ with some $\nu_1 > 0$, and $T > 0$ be an arbitrary finite number. Then for all input data of the class described in the Definition, and under the conditions for the parameters described in the Definition, there exists at least one weak solution to the Problem A.

Remark. During the proof of the Theorem, the properties of solutions, described briefly in the Definition, are refined. For example, we prove such properties as

$$\int\limits_{Q_T} \rho_i^{\gamma+\zeta_1}\, d\boldsymbol{x}\, dt \leqslant C, \quad i = 1, \ldots, N,$$

where $\zeta_1 \leqslant \dfrac{2\gamma}{3} - 1$, if $\gamma < 6$, and $\zeta_1 < \dfrac{\gamma}{2}$, if $\gamma \geqslant 6$, where C depends only on the input data of the Problem A;

$$\rho_i \in C([0,T]; L_{\zeta_2}(\Omega)), \qquad \forall \zeta_2 < \gamma, \quad i = 1, \ldots, N. \tag{14}$$

Construction of approximate solutions

Let us replace the functions ρ_{0i} by smooth functions $\rho_{0i\delta} \in C^{2+\nu_2}(\overline{\Omega})$, $0 < \nu_2 < 1$, such that

$$\delta \leqslant \rho_{0i\delta} \leqslant \delta^{-\frac{1}{\beta}}, \quad \nabla \rho_{0i\delta} \cdot \boldsymbol{n}|_{\partial\Omega} = 0,$$
$$\rho_{0i\delta} \xrightarrow[\delta \to 0]{} \rho_{0i} \quad \text{strongly in} \quad L_\gamma(\Omega), \quad i = 1, \ldots, N, \tag{15}$$

where $\boldsymbol{n}$ in the external normal vector to the boundary $\partial\Omega$ of the domain Ω, $\delta \in (0, 1]$ is a small parameter (which will be tend to zero later), and the exponent

$$\beta > \max\{\gamma, 6\} \tag{16}$$

is chosen arbitrarily and will rest fixed.

Construction of approximate solutions

We look for the approximate solution of the Problem A as the solution to the following problem (we still omit the indices m, ε and δ for the values which depend on them):

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{v}) = \varepsilon \Delta \rho_i,$$
$$\rho_i|_{t=0} = \rho_{0i}, \quad \nabla \rho_i \cdot \boldsymbol{n}|_{\partial\Omega \times (0,T)} = 0, \quad i = 1, \ldots, N, \tag{17}$$

$$\int\limits_{Q_T} \Big(\rho_i \boldsymbol{u}_i \cdot \frac{\partial \boldsymbol{\varphi}_i}{\partial t} + (\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) : (\nabla \otimes \boldsymbol{\varphi}_i) + \widetilde{p}(\rho) \operatorname{div} \boldsymbol{\varphi}_i + \rho_i \boldsymbol{f}_i \cdot \boldsymbol{\varphi}_i \Big) \, d\boldsymbol{x} \, dt =$$
$$= \int\limits_{Q_T} \Big(\varepsilon ((\nabla \otimes \boldsymbol{u}_i) \boldsymbol{\varphi}_i) \cdot \nabla \rho_i + \mathbb{S}_i : (\nabla \otimes \boldsymbol{\varphi}_i) \Big) \, d\boldsymbol{x} \, dt -$$
$$- \int\limits_{\Omega} \rho_{0i} \boldsymbol{u}_{0i} \cdot \boldsymbol{\varphi}_i(0, \boldsymbol{x}) \, d\boldsymbol{x}, \quad i = 1, \ldots, N. \tag{18}$$

Construction of approximate solutions

The integral identities (18) are supposed to take place for all $\boldsymbol{\varphi}_i \in C_0^1([0,T); X_m)$, $i = 1, \dots, N$.

Here we denote:

- $X_m = \mathrm{Lin}\,\{\boldsymbol{\psi_i}\}_{i=1}^m \subset L_2(\Omega)$, where $\{\boldsymbol{\psi_i}\}_{i=1}^m$ is a basis in $W_2^1(\Omega)$, which is orthonormal in $L_2(\Omega)$ and consists of smooth compactly supported (in Ω) functions; the norm in X_m is set to coincide with the norm in $L_2(\Omega)$;
- $\varepsilon \in (0,1]$ is a small parameter (which will be tend to zero later),
- $m \in \mathbb{N}$ (later $m \to +\infty$),
- $\widetilde{p}(s) = p(s) + \delta s^\beta$.

Construction of approximate solutions

It is well-known from the parabolic theory that if $\boldsymbol{v} \in C([0,T]; X_m)$ is given then

- there exist unique classic solutions to (17), i. e. $\rho_i \in V_{[0,T]}$, $i = 1, \ldots, N$, where

$$V_{[0,T]} = \Big\{ g \,\Big|\, \quad g \in C([0,T]; C^{2+\nu_2}(\overline{\Omega})), \quad \frac{\partial g}{\partial t} \in C([0,T]; C^{\nu_2}(\overline{\Omega})) \Big\};$$

- the mappings $\mathcal{S}_i : \boldsymbol{v} \mapsto \rho_i$, $i = 1, \ldots, N$, are bounded from $C([0,T]; X_m)$ to $V_{[0,T]}$ and are continuous with the values in $C^1([0,T] \times \overline{\Omega})$;
- for all $t \in [0,T]$, $\boldsymbol{x} \in \Omega$, $i = 1, \ldots, N$ the following estimate holds

$$\delta \exp\Big(-\|\operatorname{div} \boldsymbol{v}\|_{L_1(0,t;L_\infty(\Omega))} \Big) \leqslant \rho_i(t, \boldsymbol{x}) \leqslant \delta^{-\frac{1}{\beta}} \exp\Big(\|\operatorname{div} \boldsymbol{v}\|_{L_1(0,t;L_\infty(\Omega))} \Big); \qquad (19)$$

Construction of approximate solutions

- if $\|\boldsymbol{v}^k\|_{L_\infty(0,T;W^1_\infty(\Omega))} \leqslant \widetilde{R}$, $k=1,2$, $\widetilde{R}>0$, then for all $t\in[0,T]$ and $i=1,\ldots,N$

$$\|(S_i(\boldsymbol{v}^1)-S_i(\boldsymbol{v}^2))(t)\|_{L_2(\Omega)} \leqslant$$

$$\leqslant C(\widetilde{R},T,\varepsilon)t\|S_i(\boldsymbol{v}^{1,2})(0,\cdot)\|_{W^1_2(\Omega)}\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{L_\infty(0,T;W^1_\infty(\Omega))}. \tag{20}$$

It is not difficult to show vie the fixed point principle that there exist $\tau_0\in(0,T)$ and $\boldsymbol{u}_i\in C^1([0,\tau_0];X_m)$, $i=1,\ldots,N$, which satisfy the equations (18), in which T is replaced by τ_0, and $\rho_i=S_i(\boldsymbol{v})$, $i=1,\ldots,N$.

In order to extract the local solution to an arbitrary time interval $[0,T]$, we prove the uniform (with respect to τ_0) boundedness of the solutions $\boldsymbol{u}_i$, $i=1,\ldots,N$, to the equations (18) in the space $C([0,T];X_m)$.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Let us obtain the estimates of the solutions to the problem (17), (18), which would be uniform with respect to m, and which would be the basis for the limit with respect to $m \to +\infty$. We first note that (15) imply the inequalities

$$\delta^{1+\frac{1}{\beta}} \leqslant \frac{\rho_{0i}}{\rho_{0j}} \leqslant \delta^{-\left(1+\frac{1}{\beta}\right)}, \quad i,j = 1,\ldots,N,$$

and (17) yield that the density ratios $\dfrac{\rho_i}{\rho_j}$ satisfy the equations

$$\frac{\partial}{\partial t}\left(\frac{\rho_i}{\rho_j}\right) + \boldsymbol{v}\cdot\nabla\left(\frac{\rho_i}{\rho_j}\right) = \varepsilon\left(\Delta\left(\frac{\rho_i}{\rho_j}\right) + 2\nabla\left(\frac{\rho_i}{\rho_j}\right)\cdot\nabla(\ln\rho_j)\right), \quad i,j = 1,\ldots,N,$$

which provide the crucial relations (for all $t \in [0,T]$, $\boldsymbol{x} \in \Omega$)

$$0 < \delta^{1+\frac{1}{\beta}}\rho_j(t,\boldsymbol{x}) \leqslant \rho_i(t,\boldsymbol{x}) \leqslant \delta^{-\left(1+\frac{1}{\beta}\right)}\rho_j(t,\boldsymbol{x}), \quad i,j = 1,\ldots,N. \quad (21)$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Let us take in (18) the test functions $\boldsymbol{\varphi}_i = \chi_i \boldsymbol{u}_i$, $i = 1, \ldots, N$, where $\chi_i \in C_0^1[0, T)$, $i = 1, \ldots, N$, and we obtain

$$\int\limits_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{N} \rho_i(t) |\boldsymbol{u}_i(t)|^2 + N \widetilde{h}(\rho(t)) \right) d\boldsymbol{x} + \int\limits_{Q_t} \sum_{i=1}^{N} \mathbb{S}_i : (\nabla \otimes \boldsymbol{u}_i)\, d\boldsymbol{x}\, d\tau +$$

$$+ N\varepsilon \int\limits_{Q_t} \frac{\widetilde{p}\,'(\rho)}{\rho} |\nabla \rho|^2\, d\boldsymbol{x}\, d\tau = \int\limits_{Q_t} \sum_{i=1}^{N} \rho_i \boldsymbol{f}_i \cdot \boldsymbol{u}_i\, d\boldsymbol{x}\, d\tau +$$

$$+ \int\limits_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{N} \rho_{0i} |\boldsymbol{u}_{0i}|^2 + N \widetilde{h}(\rho_0) \right) d\boldsymbol{x} \qquad \forall\, t \in [0, T],$$

where $\widetilde{h}(s) = s \int\limits_1^s \frac{\widetilde{p}(\eta)}{\eta^2}\, d\eta = \frac{K}{\gamma - 1} \left(s^{\gamma} - s \right) + \frac{\delta}{\beta - 1} \left(s^{\beta} - s \right)$.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

This directly gives the estimate (uniform in m and ε)

$$\sup_{0\leqslant t\leqslant T}\int\limits_{\Omega}\Big(\sum_{i=1}^{N}\rho_i|\boldsymbol{u}_i|^2+\rho^{\gamma}+\rho^{\beta}\Big)\,d\boldsymbol{x}+\int\limits_{Q_T}\sum_{i=1}^{N}|\nabla\otimes\boldsymbol{u}_i|^2\,d\boldsymbol{x}\,dt+$$
$$+\varepsilon\int\limits_{Q_T}(\rho^{\gamma-2}+\rho^{\beta-2})|\nabla\rho|^2\,d\boldsymbol{x}\,dt\leqslant C. \tag{22}$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Furthermore, (22) immediately gives the estimates (uniform in m and ε)

$$\|\rho_i\|_{L_\infty(0,T;L_\beta(\Omega))} \leqslant C, \quad i = 1, \dots, N, \tag{23}$$

$$\|\sqrt{\rho_i}\boldsymbol{u}_i\|_{L_\infty(0,T;L_2(\Omega))} + \|\boldsymbol{u}_i\|_{L_2(0,T;W_2^1(\Omega))} \leqslant C, \quad i = 1, \dots, N, \tag{24}$$

and consequently, via (21), we have

$$\|\sqrt{\rho_i}\boldsymbol{v}\|_{L_\infty(0,T;L_2(\Omega))} \leqslant C, \quad i = 1, \dots, N. \tag{25}$$

Taking into account (22), we obtain the estimate of ρ_i in $L_\beta(0,T;L_{3\beta}(\Omega))$, and thus for all $\theta_1 \in [0,1]$ we obtain the estimate

$$\|\rho_i\|_{L_{\frac{\beta}{\theta_1}}\left(0,T;L_{\frac{3\beta}{3-2\theta_1}}(\Omega)\right)} \leqslant C, \quad i = 1, \dots, N, \tag{26}$$

from which (for $\theta_1 = 3/5$) we get $\|\rho_i\|_{L_{\frac{5\beta}{3}}(Q_T)} \leqslant C$, $i = 1, \dots, N$.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

From (26) as $\theta_1 = 1$ and the second estimate in (24) we have

$$\|\sqrt{\rho_i}\boldsymbol{u}_i\|_{L_{\frac{2\beta}{\beta+1}}\left(0,T;L_{\frac{6\beta}{\beta+1}}(\Omega)\right)} \leqslant C, \quad i = 1, \ldots, N,$$

and now, using the first estimate in (24), we get for all $\theta_2 \in [0, 1]$

$$\|\sqrt{\rho_i}\boldsymbol{u}_i\|_{L_{\frac{2\beta}{\theta_2(\beta+1)}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_2)\beta+\theta_2}}(\Omega)\right)} \leqslant C, \quad i = 1, \ldots, N. \tag{27}$$

Now we multiply the equations (17) by ρ_i and integrate the result over Ω, so we get for $i = 1, \ldots, N$

$$\|\sqrt{\varepsilon}\nabla\rho_i\|^2_{L_2(Q_T)} \leqslant \frac{1}{2}\Big(\|\rho_{0i}\|^2_{L_2(\Omega)} + \sqrt{T}\sum_{i=1}^{N}\|\rho_i\|^2_{L_\infty(0,T;L_4(\Omega))}\|\nabla\otimes\boldsymbol{u}_i\|_{L_2(Q_T)}\Big),$$

and after that (due to $\beta \geqslant 4$) we obtain the uniform (in m and ε) estimates

$$\|\sqrt{\varepsilon}\nabla\rho_i\|_{L_2(Q_T)} \leqslant C, \quad i = 1, \ldots, N. \tag{28}$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Basing on (23), (24) and (28), we select a subsequence (with the same notation) from the sequence $\boldsymbol{u}_{im}$, ρ_{im}, $m \in \mathbb{N}$, of the constructed solutions to the problem (17), (18), for which for all $m \to +\infty$, $i = 1, \dots, N$, the following convergences take place (below, we write the index m for the values which depend on it)

$$\rho_{im} \to \rho_i \text{ weakly* in } L_\infty(0,T;L_\gamma(\Omega)) \text{ and in } L_\infty(0,T;L_\beta(\Omega)),$$

$$\nabla\rho_{im} \to \nabla\rho_i \text{ weakly in } L_2(Q_T), \tag{29}$$

$$\boldsymbol{u}_{im} \to \boldsymbol{u}_i \text{ weakly in } L_2\big(0,T;\overset{\circ}{W}{}_2^1(\Omega)\big), \tag{30}$$

and hence

$$\boldsymbol{v}_m \to \boldsymbol{v} \text{ weakly in } L_2\big(0,T;\overset{\circ}{W}{}_2^1(\Omega)\big), \tag{31}$$

where $\boldsymbol{v} = \dfrac{1}{N}\sum\limits_{i=1}^{N}\boldsymbol{u}_i$.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Let us prove the strong convergence of the densities. The estimates (26) and (27) yield

$$\|\rho_{im}\boldsymbol{u}_{im}\|_{L_{\frac{2\beta}{\theta_2\beta+\theta_1+\theta_2}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_2)\beta+\theta_2-2\theta_1+3}}(\Omega)\right)}+ \\ +\|\rho_{im}\boldsymbol{v}_m\|_{L_{\frac{2\beta}{\theta_2\beta+\theta_1+\theta_2}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_2)\beta+\theta_2-2\theta_1+3}}(\Omega)\right)} \leqslant C \tag{32}$$

for all $i = 1, \ldots, N$, $(\theta_1, \theta_2) \in [0,1]^2$.

The equations (17), due to (28) and (32) with $\theta_1 = \theta_2 = 0$, provide

$$\left\|\frac{\partial\rho_{im}}{\partial t}\right\|_{L_2\left(0,T;W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega)\right)} \leqslant C, \quad i = 1, \ldots, N.$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Thus, the sequences ρ_{im}, $i = 1, \ldots, N$, are uniformly continuous with respect to $t \in [0, T]$ with the values in $W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega) = \left(\overset{\circ}{W}{}^{1}_{\frac{2\beta}{\beta-1}}(\Omega)\right)^*$.

Then, due to (23), we come to the convergence (here we select subsequences and preserve the notations)

$$\rho_{im} \to \rho_i \text{ as } m \to +\infty \text{ in } C([0,T]; L_{\beta,\text{weak}}(\Omega)), \quad i = 1, \ldots, N. \tag{33}$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Furthermore, due to the embeddings

$$W_2^1(\Omega) \hookrightarrow\hookrightarrow L_{\sigma_1}(\Omega) \hookrightarrow W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega) \quad \text{for all } \sigma_1 \in \left[\frac{6\beta}{5\beta+3}, 6\right),$$

the sequences $\{\rho_{im}\}$ are bounded in $L_\infty(0,T; L_{\sigma_1}(\Omega)) \bigcap L_2(0,T; W_2^1(\Omega))$, and $\left\{\dfrac{\partial \rho_{im}}{\partial t}\right\}$ are bounded in $L_2\big(0,T; W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega)\big)$, we obtain via the Loins-Aubin theorem, that for all $i = 1, \dots, N$ as $m \to +\infty$

$$\rho_{im} \to \rho_i \text{ strongly in } L_{\sigma_2}(0,T; L_{\sigma_1}(\Omega)) \quad \forall\, \sigma_2 < +\infty. \qquad (34)$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Now, using (26), we get

$$\rho_{im} \to \rho_i \text{ as } m \to +\infty \text{ strongly in } L_{\sigma_3}(0,T;L_{\sigma_4}(\Omega)), \quad i = 1,\dots,N \tag{35}$$

for all $\sigma_3 \leqslant \dfrac{\beta}{\theta_1}$, $\sigma_4 \leqslant \dfrac{3\beta}{3-2\theta_1}$, here at least one inequality must be strict.

Choosing arbitrary θ_1, θ_2 in (32), after selection of a subsequence, we may affirm the convergences

$$\rho_{im}\boldsymbol{u}_{im} \to \rho_i\boldsymbol{u}_i, \quad \rho_{im}\boldsymbol{v}_m \to \rho_i\boldsymbol{v} \quad \text{weakly* in the space (32).} \tag{36}$$

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Now, passing to the weal limit as $m \to +\infty$ in (17), we obtain that the limit functions $\boldsymbol{v}$, ρ_i, $i = 1, \ldots, N$, satisfy the equations

$$\int\limits_{Q_T} \left(\rho_i \frac{\partial \phi_i}{\partial t} + (\rho_i \boldsymbol{v} - \varepsilon \nabla \rho_i) \cdot \nabla \phi_i\right) d\boldsymbol{x} dt + \int\limits_{\Omega} \rho_{0i} \phi_i|_{t=0} \, d\boldsymbol{x} = 0$$
$$\forall\, \phi_i \in C_0^1([0,T); C^\infty(\overline{\Omega})), \quad i = 1, \ldots, N. \tag{37}$$

Let us prove that the limit functions $\boldsymbol{v}$, ρ_i, $i = 1, \ldots, N$ satisfy a. e. the equations, initial and boundary conditions (17).

Limit with respect to $m \to +\infty$ in the approximate continuity equations

The classic estimates of parabolic equations yield (due to $\beta > 6$) the uniform (in m and ε) estimates

$$\varepsilon \|\nabla \rho_{im}\|_{L_{\alpha_1}(0,T;L_{\alpha_2}(\Omega))} \leqslant C, \quad i = 1, \dots, N, \tag{38}$$

where

$$\alpha_1 = \frac{2\beta}{\theta_2\beta + \theta_1 + \theta_2}, \qquad \alpha_2 = \frac{6\beta}{(3 - 2\theta_2)\beta + \theta_2 - 2\theta_1 + 3}$$

for all $(\theta_1, \theta_2) \in [0, 1]^2 \backslash \{0, 0\}$.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Then for $i = 1, \dots, N$

$$\varepsilon \|\nabla \otimes (\rho_{im} \boldsymbol{u}_{im})\|_{L_{\alpha_3}\left(0,T;L_{\alpha_4}(\Omega)\right)} +$$

$$+\varepsilon \|\nabla \otimes (\rho_{im} \boldsymbol{v}_{m})\|_{L_{\alpha_3}\left(0,T;L_{\alpha_4}(\Omega)\right)} + \varepsilon \|(\nabla \otimes \boldsymbol{u}_{im})^* \nabla \rho_{im}\|_{L_{\alpha_3}\left(0,T;L_{\alpha_5}(\Omega)\right)} +$$

$$+\varepsilon \|(\nabla \otimes \boldsymbol{v}_{m})^* \nabla \rho_{im}\|_{L_{\alpha_3}\left(0,T;L_{\alpha_5}(\Omega)\right)} \leqslant C, \tag{39}$$

where $\alpha_3 = \dfrac{2\beta}{(\theta_2+1)\beta + \theta_1 + \theta_2}$, $\alpha_4 = \dfrac{6\beta}{2(2-\theta_2)\beta + \theta_2 - 2\theta_1 + 3}$, $\alpha_5 = \dfrac{6\beta}{3(2-\theta_2)\beta + \theta_2 - 2\theta_1 + 3}$, for all

$$\theta_1 \in [0,1], \quad \theta_2 \in \left(\frac{3-2\theta_1}{3\beta+1}, \frac{\beta-\theta_1}{\beta+1}\right) \tag{40}$$

(these restrictions provide $\alpha_{3,5} > 1$).

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Involving again classic estimates of parabolic equations, we conclude for $i = 1, \ldots, N$

$$\varepsilon^{1-\frac{1}{\alpha_3}} \|\rho_{im}\|_{L_\infty\left(0,T;W_{\alpha_4}^{2-\frac{2}{\alpha_3}}(\Omega)\right)} + \left\|\frac{\partial \rho_{im}}{\partial t}\right\|_{L_{\alpha_3}\left(0,T;L_{\alpha_4}(\Omega)\right)} + \\ +\varepsilon\|\rho_{im}\|_{L_{\alpha_3}\left(0,T;W^2_{\alpha_4}(\Omega)\right)} \leqslant C\left(\varepsilon^{1-\frac{1}{\alpha_3}}\|\rho_{0i}\|_{W_{\alpha_4}^{2-\frac{2}{\alpha_3}}(\Omega)} + \frac{1}{\varepsilon}\right). \tag{41}$$

From (41) we obtain that the limit functions ρ_i, $\boldsymbol{u}_i$, $i = 1, \ldots, N$, belong to the following classes:

$$\frac{\partial \rho_i}{\partial t} \in L_{\alpha_3}\big(0,T;L_{\alpha_4}(\Omega)\big), \qquad \rho_i \in L_{\alpha_3}\big(0,T;W^2_{\alpha_4}(\Omega)\big), \\ \nabla\otimes(\rho_i \boldsymbol{v}) \in L_{\alpha_3}\big(0,T;L_{\alpha_4}(\Omega)\big), \qquad i = 1, \ldots, N \tag{42}$$

and satisfy the equations (17) a. e. in Q_T.

Limit with respect to $m \to +\infty$ in the approximate continuity equations

Note that (42) yields

$$\rho_i \in C([0,T]; L_{\alpha_4}(\Omega)), \quad i = 1, \dots, N.$$

From (33) we may conclude that for all $t \in [0,T]$ $\rho_{im}(t) \to \rho_i(t)$ as $m \to +\infty$ weakly in $L_\beta(\Omega)$, $i = 1, \dots, N$, in particular, for $t = 0$ it allows to affirm that the initial conditions (17) are valid a. e. in Ω for the limit functions as well.

From (41) we conclude that for a. e. $t \in (0,T)$ $\rho_{im}(t) \to \rho_i(t)$ as $m \to +\infty$ weakly in $W^2_{\alpha_4}(\Omega) \hookrightarrow W^1_{\alpha_6}(\partial\Omega)$, $\alpha_6 = \dfrac{2\alpha_4}{3 - \alpha_4}$, $i = 1, \dots, N$, and since the boundary condition (17) is valid for ρ_{im}, $i = 1, \dots, N$, then $\nabla\rho_i \cdot \boldsymbol{n} = 0$ for a. e. $(t, \boldsymbol{x}) \in (0,T) \times \partial\Omega$, $i = 1, \dots, N$.

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Involving the estimate (32), we get

$$\rho_{im}\boldsymbol{u}_{im} \to \rho_i\boldsymbol{u}_i \text{ as } m \to +\infty \text{ strongly in } L_{\sigma_7}(0,T;L_{\sigma_8}(\Omega)),\ i=1,\ldots,N, \tag{43}$$

where for all $\theta_4 \in (0,1]$

$$\sigma_7 = \frac{2\beta}{(\theta_2+\theta_4)\beta+\theta_1+\theta_2}, \quad \sigma_8 = \frac{6\beta}{(3-\theta_4-2\theta_2)\beta+\theta_2-2\theta_1+3}.$$

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Now due to (30) we conclude that as $m \to +\infty$

$$\rho_{im}\boldsymbol{u}_{im}\otimes\boldsymbol{u}_{im} \to \rho_i\boldsymbol{u}_i\otimes\boldsymbol{u}_i \text{ weakly in } L_{\sigma_9}\big(0,T;L_{\sigma_{10}}(\Omega)\big), \quad i=1,\dots,N, \tag{44}$$

$$\rho_{im}\boldsymbol{u}_{im}\otimes\boldsymbol{v}_m \to \rho_i\boldsymbol{u}_i\otimes\boldsymbol{v} \text{ weakly in } L_{\sigma_9}\big(0,T;L_{\sigma_{10}}(\Omega)\big), \quad i=1,\dots,N, \tag{45}$$

where $\sigma_9 = \dfrac{2\beta}{(\theta_2+\theta_4+1)\beta+\theta_1+\theta_2}$,
$\sigma_{10} = \dfrac{6\beta}{(4-\theta_4-2\theta_2)\beta+\theta_2-2\theta_1+3}$, at that $\sigma_{10}>1$ due to the conditions introduced above, and $\sigma_9>1$ provided the inequality $\theta_4 < 1-\theta_2-\dfrac{\theta_1+\theta_2}{\beta}$, whose right-hand side is positive due to (40).

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Let us prove now the strong convergence of the gradients of the densities in $L_2(Q_T)$. From (17) we get for $i = 1, \ldots, N$

$$\frac{1}{2}\|\rho_{im}(t)\|^2_{L_2(Q_T)} + \varepsilon \int_0^T (T-t)\|\nabla \rho_{im}(t)\|^2_{L_2(\Omega)}\, dt = \\ = \frac{T}{2}\|\rho_{0i}\|^2_{L^2(\Omega)} - \frac{1}{2}\int_0^T (T-t)\int_\Omega \rho^2_{im}(t)\operatorname{div} \boldsymbol{v}_m(t)\, d\boldsymbol{x}dt. \tag{46}$$

Limit with respect to $m \to +\infty$ in the approximate momentum equations

On the other hand, for $i = 1, \dots, N$ we have similar identities for the limit functions

$$\frac{1}{2}\|\rho_i(t)\|^2_{L_2(Q_T)} + \varepsilon \int\limits_0^T (T-t)\|\nabla\rho_i(t)\|^2_{L_2(\Omega)}\,dt = \\ = \frac{T}{2}\|\rho_{0i}\|^2_{L^2(\Omega)} - \frac{1}{2}\int\limits_0^T (T-t)\int\limits_\Omega \rho_i^2(t)\mathrm{div}\,\boldsymbol{v}(t)\,d\boldsymbol{x}dt. \tag{47}$$

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Passing to the limit in (46) as $m \to +\infty$, using (31), (35) and subtracting (47) from the identities obtained, we come to the relations

$$\lim_{m\to+\infty} \int_0^T (T-t)\|\nabla\rho_{im}(t)\|^2_{L_2(\Omega)}\, dt = \int_0^T (T-t)\|\nabla\rho_i(t)\|^2_{L_2(\Omega)}\, dt, \quad i = 1,\ldots,N.$$

These equalities together with (29) show that as$m \to +\infty$

$$\nabla\rho_{im} \to \nabla\rho_i \ \text{ strongly in } \ L_2(Q_T), \quad i = 1,\ldots,N,$$

and hence, due to (38),

$$\nabla\rho_{im} \to \nabla\rho_i \ \text{ strongly in } \ L_{\sigma_{11}}(0,T;L_{\sigma_{12}}(\Omega))$$

$$\forall\, \sigma_{11} \in (2,\alpha_1)\,,\ \sigma_{12} \in (2,\alpha_2)\,, \quad i = 1,\ldots,N,$$

at that, due to (40), $\alpha_{1,2} > 2$, that yields compatibility of the conditions for $\sigma_{11,12}$.

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Using (30), we also obtain that for $m \to +\infty$ for all $i = 1, \ldots, N$

$$(\nabla \otimes \boldsymbol{u}_{im})^* \nabla \rho_{im} \to (\nabla \otimes \boldsymbol{u}_i)^* \nabla \rho_i \ \text{ weakly in } \ L_{\sigma_{13}}(0, T; L_{\sigma_{14}}(\Omega))$$

$$\forall\, \sigma_{13} \in \left(1, \frac{2\alpha_1}{2+\alpha_1}\right), \ \sigma_{14} \in \left(1, \frac{2\alpha_2}{2+\alpha_2}\right). \tag{48}$$

Now the limit in (18) as $m \to +\infty$ becomes trivial.

Limit with respect to $m \to +\infty$ in the approximate momentum equations

For all $\boldsymbol{\varphi}_i \in C_0^1([0,T); C_0^1(\Omega))$ the following equalities hold

$$\int\limits_{Q_T} \left(\rho_i \boldsymbol{u}_i \cdot \frac{\partial \boldsymbol{\varphi}_i}{\partial t} + (\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) : (\nabla \otimes \boldsymbol{\varphi}_i) + \widetilde{p}(\rho) \operatorname{div} \boldsymbol{\varphi}_i + \rho_i \boldsymbol{f}_i \cdot \boldsymbol{\varphi}_i\right) d\boldsymbol{x}\, dt =$$

$$= \int\limits_{Q_T} \Big(\varepsilon((\nabla \otimes \boldsymbol{u}_i)\boldsymbol{\varphi}_i) \cdot \nabla \rho_i + \mathbb{S}_i : (\nabla \otimes \boldsymbol{\varphi}_i)\Big)\, d\boldsymbol{x}\, dt -$$

$$- \int\limits_{\Omega} \rho_{0i} \boldsymbol{u}_{0i} \cdot \boldsymbol{\varphi}_i(0, \boldsymbol{x})\, d\boldsymbol{x}, \quad i = 1, \ldots, N.$$

Limit with respect to $m \to +\infty$ in the approximate momentum equations

Energy relations: for a. e. $t \in (0, T)$

$$\int\limits_{\Omega} \Big(\frac{1}{2} \sum_{i=1}^{N} \rho_i(t) |\boldsymbol{u}_i(t)|^2 + N \widetilde{h}(\rho(t)) \Big)\, d\boldsymbol{x} + \\ + C_0 \int\limits_{Q_t} \sum_{i=1}^{N} |\nabla \otimes \boldsymbol{u}_i|^2 \, d\boldsymbol{x}\, d\tau + N\varepsilon\delta\beta \int\limits_{Q_t} \rho^{\beta-2} |\nabla \rho|^2 \, d\boldsymbol{x}\, d\tau \leqslant \\ \leqslant \int\limits_{Q_t} \sum_{i=1}^{N} \rho_i \boldsymbol{f}_i \cdot \boldsymbol{u}_i \, d\boldsymbol{x}\, d\tau + \int\limits_{\Omega} \Big(\frac{1}{2} \sum_{i=1}^{N} \rho_{0i} |\boldsymbol{u}_{0i}|^2 + N \widetilde{h}(\rho_0) \Big)\, d\boldsymbol{x}. \tag{49}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Let us first obtain the estimates of solutions to the problem (17), (18), which would be uniform in the small parameter ε. From the inequalities (21)–(25), (28), and the relations (29), (30) and (34) we derive, for $i = 1, \ldots, N$, the estimates (now we start to write the index ε for the values which depend on ε)

$$0 \leqslant \delta^{1+\frac{1}{\beta}} \rho_{j\varepsilon}(t, \boldsymbol{x}) \leqslant \rho_{i\varepsilon}(t, \boldsymbol{x}) \leqslant \delta^{-\left(1+\frac{1}{\beta}\right)} \rho_{j\varepsilon}(t, \boldsymbol{x}) \;\text{ for a. a. }\; (t, \boldsymbol{x}) \in Q_T, \tag{50}$$

$$\|\rho_{i\varepsilon}\|_{L_\infty(0,T;L_\beta(\Omega))} \leqslant C, \tag{51}$$

$$\|\rho_{i\varepsilon}|\boldsymbol{u}_{i\varepsilon}|^2\|_{L_\infty(0,T;L_1(\Omega))} + \|\rho_{i\varepsilon}|\boldsymbol{v}_\varepsilon|^2\|_{L_\infty(0,T;L_1(\Omega))} + \\ + \|\boldsymbol{u}_{i\varepsilon}\|_{L_2(0,T;W_2^1(\Omega))} \leqslant C, \tag{52}$$

$$\sqrt{\varepsilon}\, \|\nabla \rho_{i\varepsilon}\|_{L_2(Q_T)} \leqslant C. \tag{53}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

From (51) and (52) we derive, for all $i = 1, \dots, N$, the estimates

$$\begin{gathered}\|\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}\|_{L_2\left(0,T;L_{\frac{6\beta}{\beta+6}}(\Omega)\right)} + \|\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}\|_{L_\infty\left(0,T;L_{\frac{2\beta}{\beta+1}}(\Omega)\right)} + \\ +\|\rho_{i\varepsilon}\boldsymbol{v}_{\varepsilon}\|_{L_2\left(0,T;L_{\frac{6\beta}{\beta+6}}(\Omega)\right)} + \|\rho_{i\varepsilon}\boldsymbol{v}_{\varepsilon}\|_{L_\infty\left(0,T;L_{\frac{2\beta}{\beta+1}}(\Omega)\right)} \leqslant C,\end{gathered} \tag{54}$$

from which we obtain, for all $\theta_5 \in [0,1]$, the inequalities

$$\begin{gathered}\|\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}\|_{L_{\frac{2}{\theta_5}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} + \\ +\|\rho_{i\varepsilon}\boldsymbol{v}_{\varepsilon}\|_{L_{\frac{2}{\theta_5}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} \leqslant C, \quad i = 1, \dots, N.\end{gathered} \tag{55}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Then the equations (17) provide, for all $i = 1, \dots, N$, the uniform (in ε) estimates

$$\varepsilon \|\nabla \rho_{i\varepsilon}\|_{L_{\frac{2}{\theta_5}}\left(0,T;L_{\frac{6\beta}{(3-2\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} \leqslant C \tag{56}$$

for all $\theta_5 \in (0, 1]$. Hence for all $i = 1, \dots, N$ the following inequalities hold

$$\begin{aligned} &\varepsilon \|\nabla \otimes (\rho_{i\varepsilon} \boldsymbol{u}_{i\varepsilon})\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} + \\ &+\varepsilon \|\nabla \otimes (\rho_{i\varepsilon} \boldsymbol{v}_{\varepsilon})\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} + \\ &+\varepsilon \|(\nabla \otimes \boldsymbol{u}_{i\varepsilon})^* \nabla \rho_{i\varepsilon}\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} + \\ &+\varepsilon \|(\nabla \otimes \boldsymbol{v}_{\varepsilon})^* \nabla \rho_{i\varepsilon}\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} \leqslant C. \end{aligned} \tag{57}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Finally, from (52) and (55) we derive, for all $i, j = 1, \ldots, N$, the estimates

$$\|\rho_{i\varepsilon} \boldsymbol{u}_{i\varepsilon} \otimes \boldsymbol{u}_{j\varepsilon}\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\right)} \leqslant C. \tag{58}$$

Furthermore, using the properties of Bogovskii operator, we refine the integrability of the densities:

$$\int\limits_{Q_T} \rho_{i\varepsilon}^{\beta+1}\, d\boldsymbol{x}\, dt \leqslant C, \quad i = 1, \ldots, N. \tag{59}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Due to the estimates (51), (52), (53) and (59), we may select a sequence (with the same notation) from the family $\boldsymbol{u}_{i\varepsilon}$, $\rho_{i\varepsilon}$, $\varepsilon \in (0,1]$, for which as $\varepsilon \to 0$ for all $i = 1, \ldots, N$, the following convergences take place

$$\rho_{i\varepsilon} \to \rho_i \quad * -\text{weakly in } \; L_\infty(0,T; L_\beta(\Omega)), \tag{60}$$

$$\boldsymbol{u}_{i\varepsilon} \to \boldsymbol{u}_i \; \text{ weakly in } \; L_2\big(0,T; \overset{\circ}{W}{}^1_2(\Omega)\big), \tag{61}$$

$$\varepsilon \nabla \rho_{i\varepsilon} \to 0 \; \text{ strongly in } \; L_2(Q_T), \tag{62}$$

$$\rho_{i\varepsilon} \to \rho_i \; \text{ weakly in } \; L_{\beta+1}(Q_T), \tag{63}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

$$\rho_{i\varepsilon}^{\beta} \to \overline{\rho_i^{\beta}}, \;\; \widetilde{p}(\rho_\varepsilon) \to \overline{\widetilde{p}(\rho)} \text{ weakly in } L_{\frac{\beta+1}{\beta}}(Q_T), \;\; \overline{\rho_i^{\beta}} \geqslant 0 \text{ a. e. in } Q_T \tag{64}$$

and

$$\rho_{i\varepsilon}^{\gamma} \to \overline{\rho_i^{\gamma}} \text{ weakly in } L_{\frac{\beta+1}{\gamma}}(Q_T), \;\; \overline{\rho_i^{\gamma}} \geqslant 0 \text{ a. e. in } Q_T,$$

where $\overline{\rho_i^{\beta}}$, $\overline{\rho_i^{\gamma}}$, $i = 1, \ldots, N$, and $\overline{\widetilde{p}(\rho)}$ denote weak limits of the sequences $\rho_{i\varepsilon}^{\beta}$, $\rho_{i\varepsilon}^{\gamma}$, $i = 1, \ldots, N$, and $\widetilde{p}(\rho_\varepsilon)$.

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

From the equations (17) (for the functions $\boldsymbol{v}_\varepsilon$, $\rho_{i\varepsilon}$, $i = 1, \dots, N$) due to (53) and (54) we derive for $i = 1, \dots, N$

$$\left\| \frac{\partial \rho_{i\varepsilon}}{\partial t} \right\|_{L_2\left(0,T; W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega)\right)} \leqslant C.$$

Thus, the sequances $\rho_{i\varepsilon}$, $i = 1, \dots, N$, are uniformly continuous with respect to $t \in [0, T]$ with the values in $W^{-1}_{\frac{2\beta}{\beta+1}}(\Omega) = \left(\overset{\circ}{W}{}^{1}_{\frac{2\beta}{\beta-1}}(\Omega)\right)^*$. Then, due to (33) and (51), we come to the convergence (selecting subsequences and preserving the notations)

$$\rho_{i\varepsilon} \to \rho_i \text{ as } \varepsilon \to 0 \text{ in } C([0, T]; L_{\beta,\mathrm{weak}}(\Omega)), \quad i = 1, \dots, N. \tag{65}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Since the embedding of $L_\beta(\Omega)$ into $W_2^{-1}(\Omega)$ is compact then

$$\rho_{i\varepsilon} \to \rho_i \text{ as } \varepsilon \to 0 \text{ in } L_p\big(0,T;W_2^{-1}(\Omega)\big) \quad \forall p \in [1,+\infty), \quad i=1,\ldots,N.$$

Selecting arbitrary θ_5 in (55), we may affirm (after the selection of a subsequence) that

$$\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \to \rho_i\boldsymbol{u}_i, \quad \rho_{i\varepsilon}\boldsymbol{v}_\varepsilon \to \rho_i\boldsymbol{v} \quad \text{weakly in the space (55)}, \quad i=1,\ldots,N. \tag{66}$$

Now we get from (17) that the limit functions $\boldsymbol{v}$, ρ_i, $i=1,\ldots,N$, satisfy the equations

$$\int\limits_{Q_T}\Big(\rho_i\frac{\partial\phi_i}{\partial t} + \rho_i\boldsymbol{v}\cdot\nabla\phi_i\Big)\,d\boldsymbol{x}dt + \int\limits_\Omega \rho_{0i}\phi_i|_{t=0}\,d\boldsymbol{x} = 0$$
$$\forall\,\phi_i \in C_0^1\big([0,T);C^\infty(\overline{\Omega})\big),\ \ i=1,\ldots,N \tag{67}$$

(the weak form of (6)), which mean, due to (65), that ρ_i satisfy the initial conditions in (17) in the sense of the space $C([0,T];L_{\beta,\text{weak}}(\Omega))$.

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

From the equations (18) (for the functions $\boldsymbol{u}_{i\varepsilon}$, $\rho_{i\varepsilon}$, $i = 1, \dots, N$) due to (52), (57), (58) and (59), we derive for $i = 1, \dots, N$ that

$$\left\| \frac{\partial(\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon})}{\partial t} \right\|_{L_{\frac{2}{\theta_5+1}}\left(0,T;W^{-1}_{\frac{\beta+1}{\beta}}(\Omega)\right)} \leqslant C.$$

Thus, the sequences $\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}$, $i = 1, \dots, N$, are uniformly continuous with respect to $t \in [0, T]$ with the values in $W^{-1}_{\frac{\beta+1}{\beta}}(\Omega) = \left(\overset{\circ}{W}{}^1_{\beta+1}(\Omega)\right)^*$. Then we come to the convergence (after the selection of a subsequence and preserving the notations)

$$\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \to \rho_i\boldsymbol{u}_i \ \text{ as } \ \varepsilon \to 0 \ \text{ in } \ C([0,T]; L_{\frac{2\beta}{\beta+1},\mathrm{weak}}(\Omega)), \quad i = 1, \dots, N. \tag{68}$$

Limit with respect to $\varepsilon \to 0$, except the terms with the pressure

Since the embedding of $L_{\frac{2\beta}{\beta+1}}(\Omega)$ into $W_2^{-1}(\Omega)$ is compact, then due to the estimate (58) we obtain

$$\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \otimes \boldsymbol{u}_{j\varepsilon} \to \rho_i \boldsymbol{u}_i \otimes \boldsymbol{u}_j \quad \text{as} \quad \varepsilon \to 0$$
$$\text{weakly in} \quad L_{\frac{2}{\theta_5+1}}\big(0,T; L_{\frac{6\beta}{2(2-\theta_5)\beta+3(\theta_5+1)}}(\Omega)\big), \quad i,j = 1,\dots,N. \tag{69}$$

Now, involving (61) and (62), we may pass to the limit in (18) as $\varepsilon \to 0$ and to obtain

$$\int\limits_{Q_T} \Big(\rho_i \boldsymbol{u}_i \cdot \frac{\partial \boldsymbol{\varphi}_i}{\partial t} + (\rho_i \boldsymbol{u}_i \otimes \boldsymbol{v}) : (\nabla \otimes \boldsymbol{\varphi}_i) + \overline{\widetilde{p}(\rho)} \operatorname{div} \boldsymbol{\varphi}_i + \rho_i \boldsymbol{f}_i \cdot \boldsymbol{\varphi}_i\Big)\, d\boldsymbol{x}\, dt =$$
$$= \int\limits_{Q_T} \mathbb{S}_i : (\nabla \otimes \boldsymbol{\varphi}_i)\, d\boldsymbol{x}\, dt - \int\limits_{\Omega} \rho_{0i}\boldsymbol{u}_{0i} \cdot \boldsymbol{\varphi}_i(0,\boldsymbol{x})\, d\boldsymbol{x} \tag{70}$$

for all $\boldsymbol{\varphi}_i \in C_0^1([0,T); C_0^\infty(\Omega))$, $i = 1,\dots,N$.

Conclusion of the limit with respect to $\varepsilon \to 0$

Thus, in order to finalize the limit with respect to ε, it remains to prove that

$$\overline{\widetilde{p}(\rho)} = \widetilde{p}(\rho) \text{ a. e. in } Q_T. \tag{71}$$

Let us consider, for all $i = 1, \dots, N$, the so called effective viscous fluxes of the constituents of the multifluid $\widetilde{p}(\rho) - \sum_{j=1}^{N} \nu_{ij} \mathrm{div} \boldsymbol{u}_j$, the corresponding values for the regularized problem $\widetilde{p}(\rho_\varepsilon) - \sum_{j=1}^{N} \nu_{ij} \mathrm{div} \boldsymbol{u}_{j\varepsilon}$, and their weak limits in $L_{\frac{\beta+1}{\beta}}(Q_T)$:

$$\overline{\widetilde{p}(\rho)} - \sum_{j=1}^{N} \nu_{ij} \mathrm{div} \boldsymbol{u}_j.$$

Conclusion of the limit with respect to $\varepsilon \to 0$

We are going to use the operator Δ^{-1} which acts via the formula

$$(\Delta^{-1}v)(\boldsymbol{x}) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\frac{v(\boldsymbol{y})\,d\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|},$$

and to apply it to the functions $v \in L_{\sigma_{16}}(\Omega)$, $\sigma_{16} > \dfrac{3}{2}$, which are extended as zero outside Ω. With that, $\Delta^{-1}:\ L_{\sigma_{16}}(\Omega) \to W^2_{\sigma_{16}}(\Omega)$, and $\Delta \circ \Delta^{-1} = I$.

From the equations (17) (for the functions $\boldsymbol{v}_\varepsilon$, $\rho_{i\varepsilon}$, $i = 1, \ldots, N$), after elementary transformations (which are valid due to the restriction (40) and analogues of the estimates (39) and (41) after the limit as $m \to +\infty$), we come to the identities

$$\frac{\partial \boldsymbol{r}_{j\varepsilon}}{\partial t} = -\nabla \mathrm{div}\Delta^{-1}(\rho_{j\varepsilon}\boldsymbol{v}_\varepsilon) + \varepsilon\nabla\rho_{j\varepsilon}, \quad j = 1, \ldots, N, \tag{72}$$

in which we used the notations $\boldsymbol{r}_{j\varepsilon} = \nabla\Delta^{-1}\rho_{j\varepsilon}$, $j = 1, \ldots, N$.

Conclusion of the limit with respect to $\varepsilon \to 0$

Let us take in the equations (18) (for the functions $\boldsymbol{u}_{i\varepsilon}$, $\rho_{i\varepsilon}$, $i = 1, \ldots, N$) the vector fields $\boldsymbol{\varphi}_i = \psi\tau\boldsymbol{r}_{j\varepsilon}$, $i, j = 1, \ldots, N$, as the test functions, where

$$\psi \in C_0^\infty(0,T), \qquad \tau \in C_0^\infty(\Omega). \tag{73}$$

Then, taking into account (72), we come, for all $i, j = 1, \ldots, N$, to the equalities

$$\begin{aligned}
&\int\limits_{Q_T} \psi\left(\tau\widetilde{p}(\rho_\varepsilon)\rho_{j\varepsilon} - \mathbb{S}_{i\varepsilon} : (\nabla \otimes (\tau\boldsymbol{r}_{j\varepsilon}))\right) d\boldsymbol{x}dt = \\
&= -\int\limits_{Q_T} \psi\widetilde{p}(\rho_\varepsilon)\nabla\tau \cdot \boldsymbol{r}_{j\varepsilon}\, d\boldsymbol{x}dt - \varepsilon \int\limits_{Q_T} \psi\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \cdot \nabla\rho_{j\varepsilon}\, d\boldsymbol{x}dt - \\
&\quad - \int\limits_{Q_T} \psi\tau(\rho_{i\varepsilon}\boldsymbol{v}_\varepsilon \otimes \boldsymbol{u}_{i\varepsilon}) : (\nabla \otimes \boldsymbol{r}_{j\varepsilon})\, d\boldsymbol{x}dt + \\
&\quad + \int\limits_{Q_T} \psi\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \cdot \nabla \mathrm{div}\Delta^{-1}(\rho_{j\varepsilon}\boldsymbol{v}_\varepsilon)\, d\boldsymbol{x}dt -
\end{aligned} \tag{74}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

$$-\int\limits_{Q_T} \psi(\rho_{i\varepsilon}\boldsymbol{v}_\varepsilon \otimes \boldsymbol{u}_{i\varepsilon}) : (\nabla\tau \otimes \boldsymbol{r}_{j\varepsilon})\, d\boldsymbol{x}dt - \int\limits_{Q_T} \psi\tau\rho_{i\varepsilon}\boldsymbol{f}_i \cdot \boldsymbol{r}_{j\varepsilon}\, d\boldsymbol{x}dt -$$

$$-\int\limits_{Q_T} \frac{d\psi}{dt}\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon} \cdot \boldsymbol{r}_{j\varepsilon}\, d\boldsymbol{x}dt + \varepsilon\int\limits_{Q_T} \psi\tau(\nabla \otimes \boldsymbol{u}_{i\varepsilon})^*\nabla\rho_{i\varepsilon} \cdot \boldsymbol{r}_{j\varepsilon}\, d\boldsymbol{x}dt.$$

Note that from (65) and the compactness of the embedding of $W^1_\beta(\Omega)$ into $C(\overline{\Omega})$, it follows that for $\varepsilon \to 0$ we have

$$\boldsymbol{r}_{j\varepsilon} \to \boldsymbol{r}_j \text{ in } C(\overline{Q}_T), \quad j = 1, \ldots, N. \tag{75}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

On the other hand, taking in (70) the vector fields $\boldsymbol{\varphi}_i = \psi\tau\nabla\Delta^{-1}\rho_j$, $i, j = 1, \ldots, N$, as the test functions, we derive the identities

$$\int\limits_{Q_T} \psi\left(\tau\overline{\widetilde{p}(\rho)}\rho_j - \mathbb{S}_i : (\nabla \otimes (\tau \boldsymbol{r}_j))\right) d\boldsymbol{x}dt =$$
$$= -\int\limits_{Q_T} \psi\overline{\widetilde{p}(\rho)}\nabla\tau \cdot \boldsymbol{r}_j \, d\boldsymbol{x}dt - \int\limits_{Q_T} \psi\tau(\rho_i\boldsymbol{v} \otimes \boldsymbol{u}_i) : (\nabla \otimes \boldsymbol{r}_j) \, d\boldsymbol{x}dt +$$
$$+ \int\limits_{Q_T} \psi\tau\rho_i\boldsymbol{u}_i \cdot \nabla\mathrm{div}\Delta^{-1}(\rho_j\boldsymbol{v}) \, d\boldsymbol{x}dt - \int\limits_{Q_T} \psi(\rho_i\boldsymbol{v} \otimes \boldsymbol{u}_i) : (\nabla\tau \otimes \boldsymbol{r}_j) \, d\boldsymbol{x}dt -$$
$$- \int\limits_{Q_T} \psi\tau\rho_i\boldsymbol{f}_i \cdot \boldsymbol{r}_j \, d\boldsymbol{x}dt - \int\limits_{Q_T} \frac{d\psi}{dt}\tau\rho_i\boldsymbol{u}_i \cdot \boldsymbol{r}_j \, d\boldsymbol{x}dt, \quad i, j = 1, \ldots, N, \tag{76}$$

where $\boldsymbol{r}_j = \nabla\Delta^{-1}\rho_j$, $j = 1, \ldots, N$.

Conclusion of the limit with respect to $\varepsilon \to 0$

Subtracting (76) from (74) and passing to the limit as $\varepsilon \to 0$, we obtain, due to (52), (54), (58), (59), (62)–(64), (66), (69) and (75), the relations (for $i, j = 1, \ldots, N$)

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} \psi\left(\tau\widetilde{p}(\rho_\varepsilon)\rho_{j\varepsilon} - \mathbb{S}_{i\varepsilon} : (\nabla\otimes(\tau\boldsymbol{r}_{j\varepsilon}))\right) d\boldsymbol{x}dt - \\ - \int\limits_{Q_T} \psi\left(\tau\overline{\widetilde{p}(\rho)}\rho_j - \mathbb{S}_i : (\nabla\otimes(\tau\boldsymbol{r}_j))\right) d\boldsymbol{x}dt = \\ = \lim_{\varepsilon\to 0}\int\limits_{Q_T} \psi\tau\Big(\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}\cdot\nabla\mathrm{div}\Delta^{-1}(\rho_{j\varepsilon}\boldsymbol{v}_\varepsilon) - (\rho_{i\varepsilon}\boldsymbol{v}_\varepsilon\otimes\boldsymbol{u}_{i\varepsilon}) : (\nabla\otimes\boldsymbol{r}_{j\varepsilon})\Big) d\boldsymbol{x}dt - \\ - \int\limits_{Q_T} \psi\tau\Big(\rho_i\boldsymbol{u}_i\cdot\nabla\mathrm{div}\Delta^{-1}(\rho_j\boldsymbol{v}) - (\rho_i\boldsymbol{v}\otimes\boldsymbol{u}_i) : (\nabla\otimes\boldsymbol{r}_j)\Big) d\boldsymbol{x}dt. \tag{77}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

Let us analyse the right-hand side of (77) (we are going to prove that it equals zero). Consider the operator Comm which acts as

$$\mathrm{Comm}(z,\tau) = (\nabla \otimes \nabla \Delta^{-1} z)\tau - z(\nabla \otimes \nabla \Delta^{-1} \tau),$$

and whci is known to possess the following properties: if $z_k \stackrel{w}{\to} z$ in $L_{\sigma_{17}}(\Omega)$, $\tau_k \stackrel{w}{\to} \tau$ in $L_{\sigma_{18}}(\Omega)$, where $\sigma_{17}^{-1} + \sigma_{18}^{-1} < 1$, then $\mathrm{Comm}(z_k, \tau_k) \stackrel{w}{\to} \mathrm{Comm}(z,\tau)$ in $L_{\sigma_{19}}(\Omega)$, where $\sigma_{19}^{-1} = \sigma_{17}^{-1} + \sigma_{18}^{-1}$.

Conclusion of the limit with respect to $\varepsilon \to 0$

Let us rewrite the right-hand side of (77) in the form

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T}\psi\Big(\rho_{j\varepsilon}\boldsymbol{v}_\varepsilon\cdot\nabla\mathrm{div}\Delta^{-1}(\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon})-(\tau\rho_{i\varepsilon}\boldsymbol{v}_\varepsilon\otimes\boldsymbol{u}_{i\varepsilon}):(\nabla\otimes\boldsymbol{r}_{j\varepsilon})\Big)\,d\boldsymbol{x}dt-$$
$$-\int\limits_{Q_T}\psi\Big(\rho_j\boldsymbol{v}\cdot\nabla\mathrm{div}\Delta^{-1}(\tau\rho_i\boldsymbol{u}_i)-(\tau\rho_i\boldsymbol{v}\otimes\boldsymbol{u}_i):(\nabla\otimes\boldsymbol{r}_j)\Big)\,d\boldsymbol{x}dt=$$
$$=\lim_{\varepsilon\to 0}\int\limits_{Q_T}\psi\boldsymbol{v}_\varepsilon\cdot\mathrm{Comm}(\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon},\rho_{j\varepsilon})\,d\boldsymbol{x}dt-$$
$$-\int\limits_{Q_T}\psi\boldsymbol{v}\cdot\mathrm{Comm}(\tau\rho_i\boldsymbol{u}_i,\rho_j)\,d\boldsymbol{x}dt,\quad i,j=1,\ldots,N. \tag{78}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

It follows from (65) and (68) that for all $t \in [0, T]$, $i = 1, \ldots, N$

$$\rho_{i\varepsilon}(t) \to \rho_i(t) \text{ weakly in } L_\beta(\Omega), \quad \rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}(t) \to \rho_i\boldsymbol{u}_i(t) \text{ weakly in } L_{\frac{2\beta}{\beta+1}}(\Omega),$$

and consequently

$$\mathrm{Comm}(\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}, \rho_{j\varepsilon}) \to \mathrm{Comm}(\tau\rho_i\boldsymbol{u}_i, \rho_j) \text{ weakly in } L_{\frac{2\beta}{\beta+3}}(\Omega), \quad i, j = 1, \ldots, N$$

and since the embedding of $L_{\frac{2\beta}{\beta+3}}(\Omega)$ into $W_2^{-1}(\Omega)$ is compact, then due to (51) and (54) we conclude

$$\mathrm{Comm}(\tau\rho_{i\varepsilon}\boldsymbol{u}_{i\varepsilon}, \rho_{j\varepsilon}) \to \mathrm{Comm}(\tau\rho_i\boldsymbol{u}_i, \rho_j)$$

strongly in $L_{\sigma_{20}}(0, T; W_2^{-1}(\Omega))$ for all $\sigma_{20} < \infty$, $i, j = 1, \ldots, N$.

Conclusion of the limit with respect to $\varepsilon \to 0$

These relations together with (61) lead, for all $i, j = 1, \ldots, N$, to the equalities

$$\lim_{\varepsilon \to 0} \int_{Q_T} \psi \boldsymbol{v}_\varepsilon \cdot \mathrm{Comm}(\tau \rho_{i\varepsilon} \boldsymbol{u}_{i\varepsilon}, \rho_{j\varepsilon}) \, d\boldsymbol{x} dt = \int_{Q_T} \psi \boldsymbol{v} \cdot \mathrm{Comm}(\tau \rho_i \boldsymbol{u}_i, \rho_j) \, d\boldsymbol{x} dt.$$

Thus, it follows from (77) and (78), that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{Q_T} \psi \left(\tau \widetilde{p}(\rho_\varepsilon) \rho_{j\varepsilon} - \mathbb{S}_{i\varepsilon} : (\nabla \otimes (\tau \boldsymbol{r}_{j\varepsilon})) \right) d\boldsymbol{x} dt = \\ &= \int_{Q_T} \psi \left(\tau \overline{\widetilde{p}(\rho)} \rho_j - \mathbb{S}_i : (\nabla \otimes (\tau \boldsymbol{r}_j)) \right) d\boldsymbol{x} dt, \quad i, j = 1, \ldots, N. \end{aligned} \tag{79}$$

Finally, since

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} \psi \mathbb{S}_{i\varepsilon} : (\nabla \otimes (\tau \boldsymbol{r}_{j\varepsilon}))\, d\boldsymbol{x}dt - \int\limits_{Q_T} \psi \mathbb{S}_{i} : (\nabla \otimes (\tau \boldsymbol{r}_{j}))\, d\boldsymbol{x}dt =$$
$$= \lim_{\varepsilon\to 0}\sum_{k=1}^{N}\nu_{ik}\int\limits_{Q_T}\psi\tau\rho_{j\varepsilon}\mathrm{div}\,\boldsymbol{u}_{k\varepsilon}\, d\boldsymbol{x}dt - \sum_{k=1}^{N}\nu_{ik}\int\limits_{Q_T}\psi\tau\rho_{j}\mathrm{div}\,\boldsymbol{u}_{k}\, d\boldsymbol{x}dt +$$
$$+ \lim_{\varepsilon\to 0}\sum_{k=1}^{N}\nu_{ik}\int\limits_{Q_T}\psi\mathrm{div}\,\boldsymbol{u}_{k\varepsilon}(2\nabla\tau\cdot\boldsymbol{r}_{j\varepsilon} + (\Delta\tau)\Delta^{-1}\rho_{j\varepsilon})\, d\boldsymbol{x}dt -$$
$$- \sum_{k=1}^{N}\nu_{ik}\int\limits_{Q_T}\psi\mathrm{div}\,\boldsymbol{u}_{k}(2\nabla\tau\cdot\boldsymbol{r}_{j} + (\Delta\tau)\Delta^{-1}\rho_{j})\, d\boldsymbol{x}dt -$$
$$- \lim_{\varepsilon\to 0}\int\limits_{Q_T}\psi\mathbb{S}_{i\varepsilon} : (\nabla\otimes[(\nabla\tau)\Delta^{-1}\rho_{j\varepsilon}])\, d\boldsymbol{x}dt +$$
$$+ \int\limits_{Q_T}\psi\mathbb{S}_{i} : (\nabla\otimes[(\nabla\tau)\Delta^{-1}\rho_{j}])\, d\boldsymbol{x}dt, \quad i,j = 1,\dots,N, \tag{80}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

then (due to (52) and (61) the last four integrals in (80) annihilate) the equalities (79) transform into the following relations for the effective viscous fluxes of the constituents of the multifluid

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} \psi\tau\rho_{j\varepsilon}\left(\widetilde{p}(\rho_\varepsilon) - \sum_{k=1}^{N}\nu_{ik}\mathrm{div}\boldsymbol{u}_{k\varepsilon}\right) d\boldsymbol{x}dt =$$

$$= \int\limits_{Q_T} \psi\tau\rho_j\left(\overline{\widetilde{p}(\rho)} - \sum_{k=1}^{N}\nu_{ik}\mathrm{div}\boldsymbol{u}_k\right) d\boldsymbol{x}dt, \qquad i,j = 1,\ldots,N. \tag{81}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

It follows from (81) that

$$\lim_{\varepsilon \to 0} \int\limits_{Q_T} \psi \tau \rho_\varepsilon \left(\nu_0 \widetilde{p}(\rho_\varepsilon) - \operatorname{div} \boldsymbol{v}_\varepsilon \right) \, d\boldsymbol{x} dt = \int\limits_{Q_T} \psi \tau \rho \left(\nu_0 \overline{\widetilde{p}(\rho)} - \operatorname{div} \boldsymbol{v} \right) \, d\boldsymbol{x} dt, \tag{82}$$

where $\nu_0 = \dfrac{\mathbf{N}^{-1} : \mathbf{J}}{N} > 0$, and $\mathbf{J}$ is the $N \times N$-matrix, all entries of which are equal to 1.

Conclusion of the limit with respect to $\varepsilon \to 0$

Since ψ and τ are arbitrary, the relation (82) means that

$$\nu_0\overline{\rho\widetilde{p}(\rho)} - \overline{\rho\mathrm{div}\boldsymbol{v}} = \nu_0\rho\overline{\widetilde{p}(\rho)} - \rho\mathrm{div}\boldsymbol{v} \text{ a. e. in } Q_T. \qquad (83)$$

Since the renormalized equations(6) are valid, then, in particular, for the functions $\widetilde{G} \in C[0,\infty)\bigcap C^1(0,\infty)$ such that

$$\lim_{s\to 0+}\left(s\widetilde{G}'(s) - \widetilde{G}(s)\right) \in \mathbb{R}, \qquad \left|\widetilde{G}'(s)\right| \leqslant Cs^{\sigma_{21}}$$

for all $s \in (1,\infty)$ with some $\sigma_{21} \leqslant \dfrac{\beta}{2} - 1$, the following equations are valid in $D'((0,T)\times\mathbb{R}^3)$:

$$\frac{\partial\widetilde{G}(\rho)}{\partial t} + \mathrm{div}(\widetilde{G}(\rho)\boldsymbol{v}) + (\rho\widetilde{G}'(\rho) - \widetilde{G}(\rho))\mathrm{div}\boldsymbol{v} = 0,$$

from which, for $\widetilde{G}(s) = s\ln s$, it follows that (for a. a. $t \in (0,T)$) the equation

$$\int\limits_\Omega (\rho\ln\rho)(t)\,d\boldsymbol{x} - \int\limits_\Omega \rho_0\ln\rho_0\,d\boldsymbol{x} + \int\limits_{Q_t}\rho\,\mathrm{div}\boldsymbol{v}\,d\boldsymbol{x}ds = 0 \qquad (84)$$

holds.

Conclusion of the limit with respect to $\varepsilon \to 0$

On the other hand, adding together the relations (17), we get

$$\frac{\partial \rho_\varepsilon}{\partial t} + \mathrm{div}\,(\rho_\varepsilon \boldsymbol{v}_\varepsilon) = \varepsilon \Delta \rho_\varepsilon, \quad \rho_\varepsilon|_{t=0} = \rho_0 := \sum_{i=1}^{N} \rho_{0i}, \quad \nabla \rho_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega \times (0,T)} = 0. \tag{85}$$

Multiplying (85) by $\ln(\rho_\varepsilon + h) + \dfrac{\rho_\varepsilon}{\rho_\varepsilon + h}$, $h \in (0, 1]$, integrating the result over Q_t, and then making trivial estimates and passing to the limit as $h \to 0$ and $\varepsilon \to 0$, we obtain

$$\int_\Omega \overline{\rho \ln \rho}(t)\, d\boldsymbol{x} - \int_\Omega \rho_0 \ln \rho_0\, d\boldsymbol{x} + \int_{Q_t} \overline{\rho\, \mathrm{div} \boldsymbol{v}}\, d\boldsymbol{x} d\tau \leqslant 0. \tag{86}$$

Combining (84) and (86), we come to the inequality

$$\int_{Q_t} \left(\overline{\rho\, \mathrm{div} \boldsymbol{v}} - \rho\, \mathrm{div} \boldsymbol{v} \right) d\boldsymbol{x} d\tau \leqslant \int_\Omega \left((\rho \ln \rho)(t) - \overline{\rho \ln \rho}(t) \right) d\boldsymbol{x}. \tag{87}$$

Conclusion of the limit with respect to $\varepsilon \to 0$

Due to the monotonicity of the function $\widetilde{p}(\cdot)$ (remind that $\widetilde{p}'(s) = K\gamma s^{\gamma-1} + \delta\beta s^{\beta-1}$), the pointwise inequality $(\rho_\varepsilon - \rho)(\widetilde{p}(\rho_\varepsilon) - \widetilde{p}(\rho)) \geqslant 0$ holds, due to which and the formulae (63), (64), we derive

$$\lim_{\varepsilon\to 0} \int\limits_B (\widetilde{p}(\rho_\varepsilon)\rho_\varepsilon - \widetilde{p}(\rho_\varepsilon)\rho)\, d\boldsymbol{x}dt =$$

$$= \lim_{\varepsilon\to 0} \int\limits_B (\widetilde{p}(\rho_\varepsilon) - \widetilde{p}(\rho))(\rho_\varepsilon - \rho)\, d\boldsymbol{x}dt + \lim_{\varepsilon\to 0} \int\limits_B \widetilde{p}(\rho)(\rho_\varepsilon - \rho)\, d\boldsymbol{x}dt \geqslant 0,$$

where B denotes an arbitrary ball in Q_T, hence

$$\overline{\widetilde{p}(\rho)\rho} \geqslant \overline{\widetilde{p}(\rho)}\rho \text{ a. e. in } Q_T.$$

Then it follows from (83) that

$$\overline{\rho \mathrm{div} \boldsymbol{v}} - \rho \mathrm{div} \boldsymbol{v} \geqslant 0 \text{ a. e. in } Q_T.$$

Conclusion of the limit with respect to $\varepsilon \to 0$

Coming back to (87), we now obtain the relation

$$\int\limits_{\Omega} \Big((\rho \ln \rho)(t) - \overline{\rho \ln \rho}(t) \Big)\, d\boldsymbol{x} \geqslant 0,$$

from which, using the properties of the function $s \mapsto s \ln s$ (namely, its weak lower semicontinuity and strict convexity), we conclude that

$$\rho_\varepsilon \to \rho \text{ a. e. in } Q_T, \tag{88}$$

and the proof of (71) is complete.

Therefore, the functions ρ_i, $\boldsymbol{u}_i$, $i = 1, \ldots, N$, form a solution to the Problem A, in which, however, the value p in (7) is still substituted by $\widetilde{p}$.

Limit with respect to $\delta \to 0$

The limit procedure with respect to $\delta \to 0$ is based, in general, on the same ideas as were used above, but contains some more sophisticated technical details. This procedure may be found in the publications listed below.

Publications

[1] A. E. Mamontov, D. A. Prokudin, *Solubility of initial boundary value problem for the equations of polytropic motion of multicomponent viscous compressible fluids*, Siberian Electr. Math. Reports, **13** (2016), 541–583.

[2] A. E. Mamontov, D. A. Prokudin, *Solvability of unsteady equations of multicomponent viscous compressible fluids*, to appear in: Izvestiya: Mathematics, (2016).

[3] D. A. Prokudin, M. V. Krayushkina, *Solvability of steady boundary value problem for a model system of equations of barotropic motion of a mixture of viscous compressible fluids*, Journal of Applied and Industrial Mathematics, **19**:3 (2016), 55–67.