The Group Behaviour Analysis of the High-Frequency Traders Based on Mean Field Games Approach

Nikolai Trusov[1,2,3,4] Lev Egorov[3]

[1]MSU, faculty of computational mathematics and cybernetics;
[2]MIPT, department of control and applied mathematics;
[3]Federal Research Center "Computer Science and Control", Russian Academy of Sciences, Moscow;
[4]Moscow Center for Fundamental and Applied Mathematics

30 September 2024.

The dynamic of the aggregated asset share price

Stock market members: *main investors*, $\underbrace{\text{high-frequency traders (HFTs)}}_{\text{professional traders and retail traders}}$.

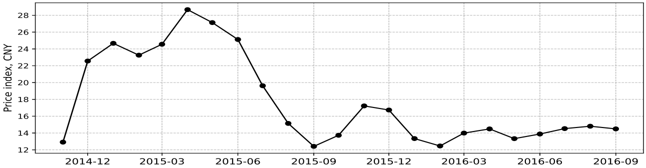


Figure: 1. Price index for 5 investment banks caused the crisis.

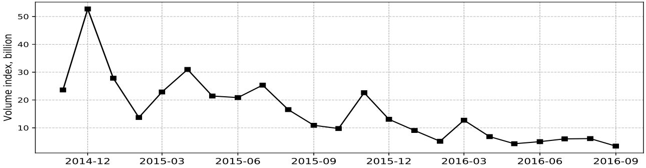


Figure: 2. Volume index for 5 investment banks caused the crisis.

$$\begin{cases} dx(t) = \alpha(t, x(t))dt + \sigma dW(t), \\ x(0) = \tilde{x}_0. \end{cases}$$

$x(t)$ — amount of asset shares held by retail trader;
$\alpha(t, x)$: $[0, T] \times \mathbb{R} \to \mathbb{R}$ — measurable control function; $\sigma > 0$;
$\tilde{x}_0$ — is a random variable with a given probability density $m_0(x)$;
$m(t, x)$: $[0, T] \times \mathbb{R} \to \mathbb{R}_+$ — is a probability density function of the retail traders by the amount of asset shares.

$$U(\alpha) = \mathbb{E}\left(\int_0^T \left(\ln(m(t, x(t))) - k\alpha^2(t, x(t)) - \lambda\left(x(t) - \tilde{a}(t)\right)^2\right) dt - \theta(x(T) - a)^2\right).$$

The goal of the retail traders is to maximize the utility function

$U(\alpha) \to \max.$

L. Fatone, F. Mariani, M. C. Recchioni, F. Zirilli. (2014) A Trading Execution Model Based on Mean Field Games and Optimal Control. Applied Mathematics, 5, P. 3091-116.

$$u(t,x) = \max_{\alpha} \mathbb{E}\left(\int_t^T \left(\ln(m(\tau, x(\tau))) - k\alpha^2(\tau, x(\tau)) - \lambda \left(x(\tau) - \tilde{a}(\tau) \right)^2 \right) d\tau - \theta \left(x(T) - a \right)^2 \right),$$

where $x(t) = x$.

Kolmogorov–Fokker–Planck equation, evolving forward in time, and a Hamilton–Jacobi–Bellman equation, evolving backwards in time with a coupling condition $\alpha(t,x) = \dfrac{1}{2k}\dfrac{\partial u}{\partial x}$:

$$\begin{cases} \frac{\partial m(t,x)}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m(t,x)}{\partial x^2} + \frac{1}{2k}\frac{\partial}{\partial x}\left(\frac{\partial u(t,x)}{\partial x} m\right) = 0, \\ \frac{\partial u(t,x)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{4k}\left(\frac{\partial u(t,x)}{\partial x}\right)^2 - \lambda \left(x - \tilde{a}(t)\right)^2 = -\ln m(t,x), \\ m(0,x) = m_0(x), \\ u(T,x) = -\theta(x-a)^2. \end{cases} \tag{1}$$

Example when the reduction of PDEs to ODEs takes place

Statement

Let $\mu_0 \in \mathbb{R}$*,* $\delta_0 > 0$*. Assume that:*

$$m_0(x) = \frac{1}{\sqrt{2\pi\delta_0^2}} \exp\left[-\frac{1}{2\delta_0^2}\left(x-\mu_0\right)^2\right], \quad x \in \mathbb{R},$$

$\tilde{x}_0 \sim N(\mu_0, \delta_0)$.
the solution of the system of PDEs (1) is given by

$$u(t,x) = C_0(t) + C_1(t)x + C_2(t)x^2, \quad x \in \mathbb{R}, t \in [0,T], \tag{2}$$

$$m(t,x) = \exp\left[D_0(t) + D_1(t)x + D_2(t)x^2\right], \quad x \in \mathbb{R}, t \in [0,T], \tag{3}$$

where the functions $C_0(t)$*,* $C_1(t)$*,* $C_2(t)$*,* $D_0(t)$*,* $D_1(t)$*,* $D_2(t)$ *satisfy the following system of Riccati-type ODEs:*

Example when the reduction of PDEs to ODEs takes place

Statement (continuation)

$$
\begin{cases}
\frac{dD_0}{dt} = -\frac{1}{2k}C_1 D_1 - \frac{1}{k}C_2 + \frac{\sigma^2}{2}D_1^2 + \sigma^2 D_2, \\
D_0(0) = -\frac{\mu_0^2}{2\delta_0^2} - \frac{1}{2}\ln\left(2\pi\delta_0^2\right), \\
\frac{dD_1}{dt} = -\frac{1}{k}C_1 D_2 - \frac{1}{k}C_2 D_1 + 2\sigma^2 D_1 D_2, \;\; D_1(0) = \frac{\mu_0}{\delta_0^2}, \\
\frac{dD_2}{dt} = -\frac{2}{k}C_2 D_2 + 2\sigma^2 D_2^2, \\
D_2(0) = -\frac{1}{2\delta_0^2}, \\
\frac{dC_0}{dt} = -\frac{1}{4k}C_1^2 - \sigma^2 C_2 - D_0 + \lambda\tilde{a}^2, \\
C_0(T) = -a^2\theta, \\
\frac{dC_1}{dt} = -\frac{1}{k}C_1 C_2 - D_1 - 2\lambda\tilde{a}, \\
C_1(T) = 2a\theta, \\
\frac{dC_2}{dt} = -\frac{1}{k}C_2^2 - D_2 + \lambda, \\
C_2(T) = -\theta.
\end{cases}
$$

L. Fatone, F. Mariani, M. C. Recchioni, F. Zirilli. (2014) A Trading Execution Model Based on Mean Field Games and Optimal Control. Applied Mathematics, 5, P. 3091-116.

$$\frac{dD_2}{dt} = -\frac{2}{k}C_2D_2 + 2\sigma^2 D_2^2, \tag{4}$$

$$D_2(0) = -\frac{1}{2\delta_0^2}, \tag{5}$$

$$\frac{dC_2}{dt} = -\frac{1}{k}C_2^2 - D_2 + \lambda, \tag{6}$$

$$C_2(T) = -\theta. \tag{7}$$

Stationary points:

- $A\left(-\sqrt{k\lambda}, 0\right)$ that is a source stationary point;
- $B\left(-\frac{1+\sqrt{1+4k\lambda\sigma^4}}{2\sigma^2}, -\frac{1+\sqrt{1+4k\lambda\sigma^4}}{2k\sigma^4}\right)$ that is a saddle stationary point;
- $C\left(\sqrt{k\lambda}, 0\right)$ that is a sink stationary point;
- $D\left(\frac{\sqrt{1+4k\lambda\sigma^4}-1}{2\sigma^2}, \frac{\sqrt{1+4k\lambda\sigma^4}-1}{2k\sigma^4}\right)$ that is a saddle stationary point.

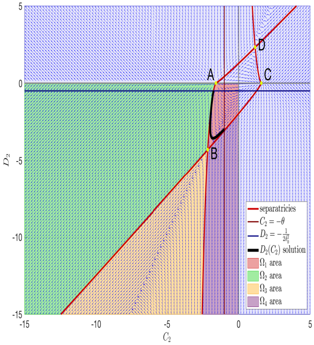


Figure: Intersection in Ω_1; $\delta_0 = 1$, $\theta = 1$, $k = 0.5$, $\lambda = 5$, $T = 0.8$.

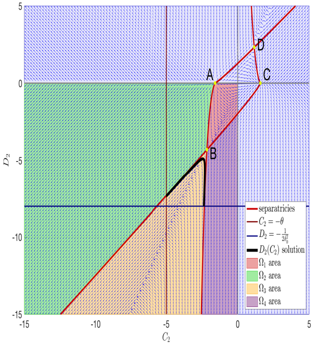


Figure: Intersection in Ω_3; $\delta_0 = 0.25$, $\theta = 5$, $k = 0.5$, $\lambda = 5$, $T = 0.5$.

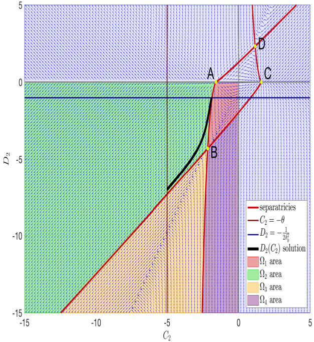


Figure: Intersection in Ω_2; $\delta_0 = 0.7$, $\theta = 5$, $k = 0.5$, $\lambda = 5$, $T = 0.5$.

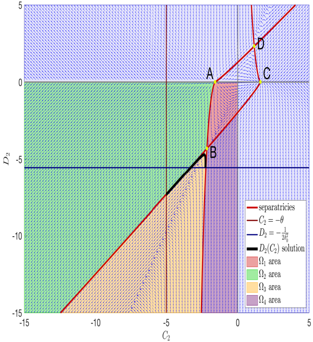


Figure: Intersection in Ω_2; $\delta_0 = 0.3$, $\theta = 5$, $k = 0.5$, $\lambda = 5$, $T = 0.5$.

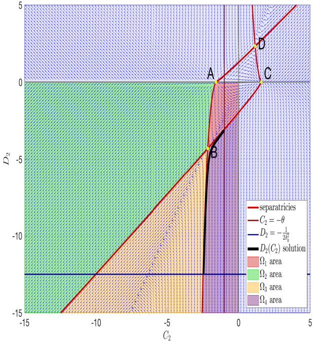


Figure: Intersection in Ω_4; $\delta_0 = 0.2$, $\theta = 1$, $k = 0.5$, $\lambda = 5$, $T = 0.5$.

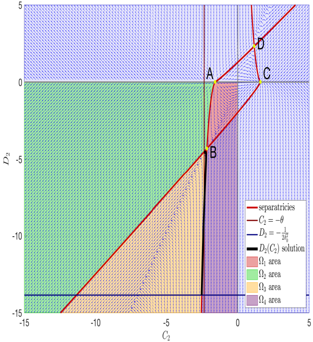


Figure: Intersection in Ω_4; $\delta_0 = 0.19$, $\theta = 2.35$, $k = 0.5$, $\lambda = 5$, $T = 0.5$.

Quadrature solution of the subsystem D_2, C_2

Proposition

The solution of the Riccati-type ODEs (4), (6) with boundary conditions (5), (7) is described by the integral curve

$$\varphi(C_2, D_2) = C_2^2 - 2\sigma^2 k D_2 C_2 - k D_2 \ln(-D_2) - k\lambda + \hat{C} k D_2, \tag{8}$$

where

$$\hat{C} = -\frac{\theta^2}{kD_2(T)} - 2\sigma^2\theta + \ln(-D_2(T)) + \frac{\lambda}{D_2(T)}. \tag{9}$$

Corollary

The solution of the Riccati-type ODEs (4), (6) with boundary conditions (5), (7) can be written in quadrature form:

- *If $C_2(t) \neq \sigma^2 k D_2(t)$, then the function $D_2(t)$ determines as*

$$\int\limits_{-\frac{1}{2\delta_0^2}}^{D_2(t)} \frac{dz}{\sqrt{(\sigma^2 kz)^2 + kz\ln(-z) + k\lambda - \hat{C}kz}} = \mp\frac{2t}{k}, \tag{10}$$

  and the function $C_2(t)$ determines as

$$C_2(t) = \sigma^2 k D_2(t) - \sqrt{(\sigma^2 k D_2(t))^2 + kD_2(t)\ln(-D_2(t)) + k\lambda - \hat{C}kD_2(t)}. \tag{11}$$

Quadrature solution of the subsystem D_2, C_2

Corollary (continuation)

The solution of the Riccati-type ODEs (4), (6) with boundary conditions (5), (7) can be written in quadrature form:

- *Otherwise, if* $C_2(t) = \sigma^2 k D_2(t)$*, then the function* $C_2(t)$ *determines as*

$$\int\limits_{C_2(t)}^{-\theta} \frac{dz}{-\frac{1}{k}z^2 - \frac{z^2 - k\lambda}{kW\left(-\frac{z^2-k\lambda}{k}\exp\left[2\sigma^2 z - \hat{C}\right]\right)} + \lambda} = T - t, \qquad (12)$$

and the function $D_2(t)$ *determines as*

$$D_2(t) = \frac{C_2^2(t) - k\lambda}{kW\left(-\frac{C_2^2(t)-k\lambda}{k}\exp\left[2\sigma^2 C_2(t) - \hat{C}\right]\right)}, \qquad (13)$$

where $\hat{C}$ *is described by the expression (9),* $C_2(t)$ *is described by (12),* W *is a Lambert* W *function.*

We assume that the asset share price $S(t)$ satisfies the ODE

$$\frac{dS}{dt} = \eta S(t)\left(2kM(t) + f(t)\right), \quad S(0) = \tilde{S}_0,$$

where $f(t) : [0, T] \to \mathbb{R}$ is the influence of the professional traders, $\eta > 0$, $M(t)$ is the influence of the retail traders,

$$M(t) = \int\limits_{\mathbb{R}} \alpha(t, x) m(t, x) dx = \frac{1}{2k}\left(C_1(t) - C_2(t)\frac{D_1(t)}{D_2(t)}\right).$$

Optimal control problem: identify the turnpike $\tilde{a}(t)$ that restores statistics $\tilde{S}(t)$.

$$\int\limits_0^T \left[\tilde{S}(t) - S(t)\right]^2 dt \to \min_{\tilde{a}}, \tag{14}$$

$$\frac{dS}{dt} = \eta S\left(C_1 - C_2\frac{D_1}{D_2} + f\right), \quad S(0) = \tilde{S}(0), \tag{15}$$

$$\frac{dC_1}{dt} = -\frac{1}{k}C_1C_2 - D_1 - 2\lambda\tilde{a}, \quad C_1(T) = 2a\theta, \tag{16}$$

$$\frac{dD_1}{dt} = -\frac{1}{k}C_1D_2 - \frac{1}{k}C_2D_1 + 2\sigma^2 D_1D_2, \quad D_1(0) = \frac{\mu_0}{\delta_0^2}. \tag{17}$$

Optimal control synthesis

Theorem

The optimal control problem (14)-(17) has a solution in the form of synthesis

$$\tilde{a} = -\frac{D_1}{2D_2} + \frac{1}{2\lambda}\frac{df}{dt} - \frac{1}{2\lambda\eta S}\cdot\frac{d^2 S}{dt^2} + \frac{1}{2\lambda S}\left[C_1 - C_2\frac{D_1}{D_2} + f\right], \tag{18}$$

when

$$\mu_0 - a + \frac{1}{2k\eta}\ln\frac{\tilde{S}(T)}{\tilde{S}_0} + \frac{1}{2\eta\theta}\frac{1}{\tilde{S}(T)}\left.\frac{d\tilde{S}(t)}{dt}\right|_{t=T} - \frac{1}{2\theta}f(T) - \frac{1}{2k}\int\limits_0^T f(t)dt = 0. \tag{19}$$

Step 1. Pontryagin maximum principle.

$$\mathcal{H}\left(S, C_1, D_1, \psi_S, \psi_{C_1}, \psi_{D_1}\right) = -\left[\tilde{S} - S\right]^2 + \psi_S S \eta \left(C_1 - C_2 \frac{D_1}{D_2} + f\right) - \\ - \psi_{C_1} \left(\frac{1}{k} C_1 C_2 + D_1 + 2\lambda \tilde{a}\right) + \psi_{D_1} \left(-\frac{1}{k} C_1 D_2 - \frac{1}{k} C_2 D_1 + 2\sigma^2 D_1 D_2\right). \quad (20)$$

The adjoint variables satisfy the system of ODEs

$$\frac{d\psi_S}{dt} = -2\left(\tilde{S} - S\right) - \eta \psi_S \left(C_1 - C_2 \frac{D_1}{D_2} + f\right), \quad \psi_S(T) = 0, \quad (21)$$

$$\frac{d\psi_{C_1}}{dt} = -\eta \psi_S S + \frac{1}{k} C_2 \psi_{C_1} + \frac{1}{k} D_2 \psi_{D_1} = 0, \quad \psi_{C_1}(0) = 0, \quad (22)$$

$$\frac{d\psi_{D_1}}{dt} = \eta \frac{C_2}{D_2} \psi_S S + \psi_{C_1} + \psi_{D_1} \left(\frac{1}{k} C_2 - 2\sigma^2 D_2\right), \quad \psi_{D_1}(T) = 0, \quad (23)$$

We construct the solution when

$$\psi_{C_1}(t) \equiv 0, \forall t \in [0, T]. \quad (24)$$

If the condition (24) is true, then an optimal control synthesis is described by (18).

Step 2. Substitute the found control (18) to the optimal control problem (14)-(17). Denote by

$$E = C_1 - \frac{C_2 D_1}{D_2}. \tag{25}$$

We obtain that

$$\frac{dC_1}{dt} = -\frac{1}{k} C_2 E + \frac{D_1}{D_2}\frac{dC_2}{dt} + \frac{1}{\eta S}\cdot\frac{d^2\tilde{S}}{dt^2} - \frac{1}{S}\cdot\frac{d\tilde{S}}{dt}(E+f) - \frac{df}{dt}, \quad C_1(T) = 2a\theta, \tag{26}$$

$$\frac{dD_1}{dt} = -\frac{1}{k} D_2 E + \frac{D_1}{D_2}\cdot\frac{dD_2}{dt}, \quad D_1(0) = \frac{\mu_0}{\delta_0^2}, \tag{27}$$

$$\frac{dS}{dt} = \eta S(E+f), \quad S(0) = \tilde{S}(0). \tag{28}$$

Differentiating (25) leads to

$$\frac{dE}{dt} = \frac{1}{\eta}\frac{d\left[\frac{1}{S}\frac{d\tilde{S}}{dt} - \eta f\right]}{dt}, \quad \text{i.e. } E = \frac{1}{\eta S}\frac{d\tilde{S}}{dt} - f + C_E, \quad \text{where } C_E \text{ is a constant.}$$

Thus, the asset price satisfies the ODE

$$\frac{dS}{dt} = \frac{d\tilde{S}}{dt} + C_E \eta S.$$

The function $S(t)$ restores statistics $\tilde{S}(t)$, $t \in [0,T]$, only when the constant $C_E = 0$, that is the optimality condition (24) is true.

Moreover,

$$D_1(t) = -D_2(t)\left[\frac{1}{k\eta}\ln\frac{\tilde{S}(t)}{\tilde{S}_0} - \frac{1}{k}\int\limits_0^t f(\tau)d\tau + 2\mu_0\right],$$

$$C_1(t) = -2\mu_0\theta - 2\mu_0 C_2(t) - \frac{1}{k\eta}\left(\theta\ln\frac{\tilde{S}(T)}{\tilde{S}_0} + C_2(t)\ln\frac{\tilde{S}(t)}{\tilde{S}_0}\right) -$$
$$-\frac{1}{\eta}\left(\frac{1}{\tilde{S}(T)}\left.\frac{d\tilde{S}}{dt}\right|_{t=T} - \frac{1}{\tilde{S}(t)}\frac{d\tilde{S}}{dt}\right) + 2a\theta + f(T) - f(t) + \frac{1}{k}\left(\theta\int\limits_0^T f(\tau)d\tau + C_2(t)\int\limits_0^t f(\tau)d\tau\right).$$

The expression

$$C_1(t) - \frac{C_2(t)D_1(t)}{D_2(t)} = \frac{1}{\eta\tilde{S}(t)}\frac{d\tilde{S}}{dt} - f(t)$$

holds true under assumption (19).

Corollary

The optimal control synthesis can be written in the form

$$\tilde{a} = \frac{1}{2k\eta}\ln\frac{\tilde{S}(t)}{\tilde{S}_0} - \frac{1}{2k}\int\limits_0^t f(\tau)d\tau + \mu_0 + \frac{1}{2\lambda}\frac{df}{dt} - \frac{1}{2\lambda\eta}\frac{d^2\left(\ln\tilde{S}(t)\right)}{dt^2}.$$

Professional traders:

$$\begin{cases} \frac{dy}{dt} = f(t), \\ \frac{dP}{dt} = -\tilde{S}(t)f(t), \\ y(0) = 0, \\ P(0) = 0. \end{cases}$$

Here $y(t)$ — represent the amount of asset shares held by the professional traders $t \in [0, T]$, $P(t)$ — determines the dynamic of the budget of the professional traders at time $t \in [0, T]$.
Retail traders:

$$\begin{cases} \frac{dz}{dt} = \frac{1}{2k}\left(\frac{1}{\eta\tilde{S}(t)}\frac{d\tilde{S}(t)}{dt} - f(t)\right) \\ \frac{dK}{dt} = -\frac{\tilde{S}(t)}{2k}\left(\frac{1}{\eta\tilde{S}(t)}\frac{d\tilde{S}(t)}{dt} - f(t)\right), \\ z(0) = \mu_0, \\ K(0) = 0. \end{cases}$$

Here $z(t)$ — the amount of asset shares of the representative retail traders at time $t \in [0, T]$, $K(t)$ — the budget of the representative retail traders at time $t \in [0, T]$.
By the results of HFTs traders during the considered time period $[0, T]$ we introduce the following characteristic. For the retail traders:

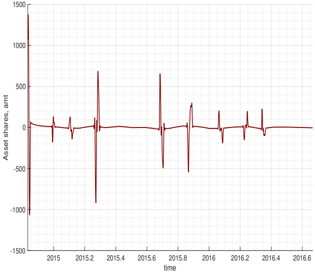
$$RT = K(T) - K(0) + z(T)\tilde{S}(T) - z(0)\tilde{S}(0),$$

and for the professional traders:

$$PT = P(T) - P(0) + y(T)\tilde{S}(T) - y(0)\tilde{S}(0).$$

Consider: $k = 0.5$, $\lambda = 5$, $\delta_0 = 0.5$, $\mu_0 = 0$, $\theta = 1$, $a = 0.31308$, $\sigma = 1$, $\eta = 1$.

(a) The strategy of the professional traders.



(b) The identification result of the retail traders' preference to hold asset shares.

Numerical examples

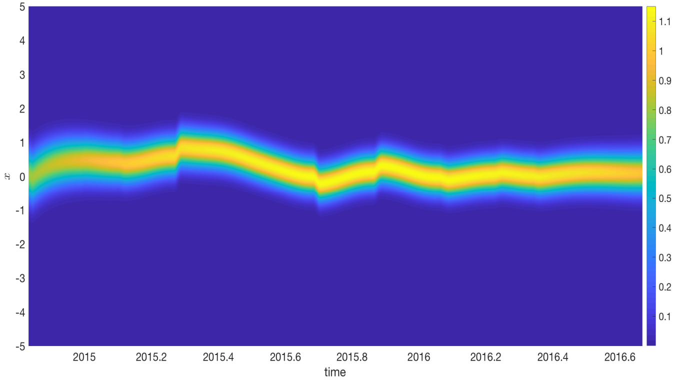
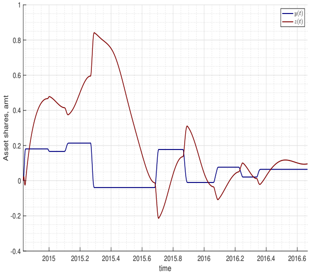
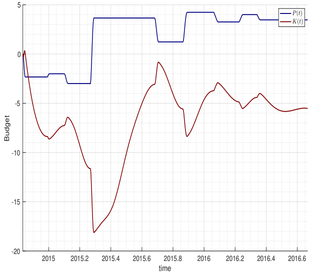


Figure: The retail traders' density function as a projection on the plane (t, x).



(a) Asset shares amount of the high-frequency traders.



(b) Budget of the high-frequency traders.

The profit of the professional traders $PT = 4.3625$ units. The retail traders did not make any profit: $RT = -4.2038$.

In monetary terms, the profit of the professional traders is 1.4384 trillion CNY, and the loss of the retail traders during stock market crisis is 1.3861 trillion CNY.

Thank you for Your attention!

L. Fatone, F. Mariani, M.C. Recchioni and F. Zirilli, A Trading Execution Model Based on Mean Field Games and Optimal Control, Appl. Math. 5 (2014), 3091-3116.

N. I. Klemashev, A. A. Shananin and S. Zhang, Inverse problems in Pareto's demand theory and their applications to analysis of stock market crises, J. Inverse Ill-Posed Probl. 26 (2018), no. 1, 95-108.

A.S. Kuts and N.V. Trusov, Analysis of the stock market crisis based on mean field games concept, Lobachevskii J. Math. 45 (2024), no. 1, 272-286.

N.V. Trusov, Numerical solution of mean field games problems with turnpike effect, Lobachevskii J. Math. 41 (2020), no. 4, 559-573.

N. Trusov, Application of mean field games approximation to economic processes modeling (in Russian), Tr. ISA RAN 68 (2018), no. 2, 88-91.

N.V. Trusov, Numerical study of the stock market crises based on mean field games approach, J. Inverse Ill-Posed Probl. 29 (2021), no. 6, 849-865.

N.V. Trusov. Numerical study of the stock market crises based on Mean Field Games approach. J. Inverse Ill-Posed (2021). https://doi.org/10.1515/jiip-2020-0016